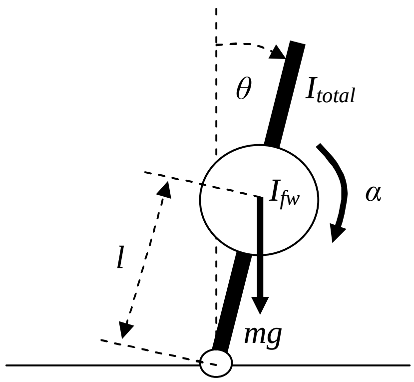
Final Examination **Control Theory AT74.02** **January 3, 2012**

Time: 9:00-11:00 hrs. Open Book
Marks: 100

Attempt all questions.

Consider the system from midterm examination of flywheel based balancing of inverted pendulum system as shown in the below figure.



After linearization, the relation between flywheel acceleration, α, and inverted pendulum leaning angle, θ, is represented by the following differential equation.

$$mgl\theta + I_{fw}\alpha = I_{total}\ddot{\theta}$$

When m = 10 kg, g = 10 m/s^2, l = 50 cm, I_{fw} = 20 kg·m^2, I_{total} = 25 kg·m^2.

(a) Determine a state-space representation of the system when the state variable x_1 represents the inverted pendulum leaning angle and the state variable x_2 represents the rate of the inverted pendulum leaning angle. Assume there is a gyro sensor used to measure the inverted pendulum leaning angle. (20)

<u>Solution</u>

$$x_1 = \theta \quad (1)$$

$$x_2 = \dot{\theta} \quad (2)$$

Thus,

$$\dot{x}_1 = x_2 \quad (3)$$

And

$$mglx_1 + I_{fw}\alpha = I_{total}\dot{x}_2 \quad (4)$$

Thus,

$$\dot{x}_2 = \frac{mgl}{I_{total}}x_1 + \frac{I_{fw}}{I_{total}}\alpha \quad (5)$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{mgl}{I_{total}} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{I_{fw}}{I_{total}}\end{bmatrix}[\alpha] \qquad (6)$$

$$[y] = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad (7)$$

(b) Design the compensator by separation method. Determine the function of flywheel acceleration that makes all the roots of the characteristic equation locate at -2. Determine the reduced-order observer gain that makes the pole of the reduced-order observer locate at -20 and also determine the equations used to estimate all the state variables. (30)

Solution

Design the regulator,

The required characteristic equation of the regulated plant is

$$(s+2)^2 = s^2 + 4s + 4 = 0 \qquad (1)$$

$$|sI - A_c| = |sI - A + BG| = \left|\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\ \frac{mgl}{I_{total}} & 0\end{bmatrix} + \begin{bmatrix}0\\ \frac{I_{fw}}{I_{total}}\end{bmatrix}[g_1 \quad g_2]\right| = 0 \qquad (2)$$

$$|sI - A_c| = \begin{vmatrix}s & -1\\ \frac{I_{fw}g_1}{I_{total}} - \frac{mgl}{I_{total}} & s + \frac{I_{fw}g_2}{I_{total}}\end{vmatrix} = 0 \qquad (3)$$

$$|sI - A_c| = s^2 + \left(\frac{I_{fw}g_2}{I_{total}}\right)s + \frac{I_{fw}g_1}{I_{total}} - \frac{mgl}{I_{total}} = s^2 + 4s + 4 \qquad (4)$$

Substitute all the parameters,

$$s^2 + \left(\frac{20g_2}{25}\right)s + \frac{20g_1}{25} - \frac{50}{25} = s^2 + 4s + 4 \qquad (5)$$

$$g_1 = 7.5 \qquad (6)$$

$$g_2 = 5 \qquad (7)$$

Thus,

$$[\alpha] = -[7.5 \quad 5]\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad (8)$$

Design the observer,

$$C_1 = [1], C_2 = [0], x_1 = [x_1], x_2 = [x_2] \qquad (9)$$

$$A_{11} = [0], A_{12} = [1], A_{21} = [2], A_{22} = [0], B_1 = [0], B_2 = [0.8] \qquad (10)$$

$$\hat{x}_1 = x_1 \qquad (11)$$

$$\hat{x}_2 = Ly + z \qquad (12)$$

$$\dot{z} = F\hat{x}_2 + \bar{g}y + Hu \qquad (13)$$

$$F = A_{22} - LC_1A_{12} = [0] - [L][1][1] = [-L] \qquad (14)$$

$$|sI - F| = [s + L] = [s + 20] \qquad (15)$$

$$[L] = [20] \qquad (16)$$

$$\bar{g} = (A_{21} - LC_1A_{11})C_1^{-1} = ([2] - [20][1][0])[1]^{-1} = [2] \quad (17)$$

$$H = B_2 - LC_1B_1 = [0.8] - [20][1][0] = [0.8] \quad (18)$$

Substitute all the concerned matrices into (12) and (13),

$$\hat{x}_2 = [20]y + z \quad \underline{(19)}$$

$$\dot{z} = [-20]\hat{x}_2 + [2]y + [0.8]\alpha \quad \underline{(20)}$$

(c) Assume the cost function is expressed by $V = \int_0^\infty \left(10\theta^2 + 5\dot{\theta}^2 + \alpha^2\right)dt$, determine the function of flywheel acceleration that minimizes the cost function. Determine also the characteristic equation and the poles of the regulated system. (25)

<u>Solution</u>

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \quad (1)$$

$$R = [1] \quad (2)$$

Control signal is determined from

$$u = -Gx = -R^{-1}B^t\bar{M}x \quad (3)$$

$$G = [1]^{-1}\begin{bmatrix} 0 \\ 0.8 \end{bmatrix}^t \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = [1][0 \quad 0.8]\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \quad (4)$$

$$G = [0.8m_2 \quad 0.8m_3] \quad (5)$$

When

$$0 = -\dot{\bar{M}} = \bar{M}A + A^t\bar{M} - \bar{M}BG + Q \quad (6)$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} - \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}\begin{bmatrix} 0 \\ 0.8 \end{bmatrix}[0.8m_2 \quad 0.8m_3] + \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \quad (7)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.64m_2^2 + 4m_2 + 10 & m_1 + 2m_3 - 0.64m_2m_3 \\ m_1 + 2m_3 - 0.64m_2m_3 & -0.64m_3^2 + 2m_2 + 5 \end{bmatrix} \quad (8)$$

Thus,

$$\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = \begin{bmatrix} 18.62 & 8.16 \\ 8.16 & 5.77 \end{bmatrix} \quad (9)$$

$$G = [6.53 \quad 4.62] \quad (10)$$

Thus,

$$[\alpha] = -[6.53 \quad 4.62]\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \quad (\underline{11})$$

Determine characteristic equation of the regulated system,

$$|sI - A_c| = |sI - A + BG| = \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}[6.53 \quad 4.62]\right| \qquad (12)$$

$$|sI - A_c| = \begin{vmatrix} s & -1 \\ 3.22 & s + 3.70 \end{vmatrix} = s^2 + 3.70s + 3.22 = 0 \qquad \underline{(13)}$$

$$s = -1.4, -2.3 \qquad \underline{(14)}$$

(d) If the inverted pendulum is disturbed by Gaussian white noise torque, *T*, with power spectral density of 25 as expressed by the equation, $mgl\theta + I_{fw}\alpha + T = I_{total}\ddot{\theta}$, and the output reading of pendulum leaning angle is contaminated by Gaussian white noise, *w*, with power spectral density of 0.0002, determine Kalman filter gain, characteristic equation of the optimal observer, and its roots. (25)

Solution

$$mgl\theta + I_{fw}\alpha + T = I_{total}\ddot{\theta} \qquad (1)$$

$$\ddot{\theta} = \frac{mgl}{I_{total}}\theta + \frac{I_{fw}}{I_{total}}\alpha + \frac{1}{I_{total}}T \qquad (2)$$

$$\dot{x}_2 = \frac{mgl}{I_{total}}x_1 + \frac{I_{fw}}{I_{total}}\alpha + \frac{1}{I_{total}}T \qquad (3)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{I_{total}} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I_{fw}}{I_{total}} \end{bmatrix}[\alpha] + \begin{bmatrix} 0 \\ \frac{1}{I_{total}} \end{bmatrix}[T] \qquad (4)$$

$$[y] = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [w] \qquad (5)$$

$$K = \bar{P}C^tW^{-1} \qquad (6)$$

$$K = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}[1 \quad 0]^t[0.0002]^{-1} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}[5000] = \begin{bmatrix} 5000p_1 \\ 5000p_2 \end{bmatrix} \qquad (7)$$

When

$$0 = -\dot{\bar{P}} = A\bar{P} + \bar{P}A^t - KC\bar{P} + FVF^t \qquad (8)$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 5000p_1 \\ 5000p_2 \end{bmatrix}[1 \quad 0]\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.04 \end{bmatrix}[25][0 \quad 0.04] \qquad (9)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -5000p_1^2 + 2p_2 & p_3 + 2p_2 - 5000p_1p_2 \\ p_3 + 2p_2 - 5000p_1p_2 & -5000p_2^2 + 4p_2 + 0.04 \end{bmatrix} \qquad (10)$$

Thus,

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0.0011 & 0.0033 \\ 0.0033 & 0.0163 \end{bmatrix} \qquad (11)$$

$$K = \begin{bmatrix} 5.5 \\ 16.5 \end{bmatrix} \qquad \underline{(12)}$$

Determine characteristic equation of the Kalman filter,

$$|sI - \hat{A}| = |sI - A + KC| = \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 5.5 \\ 16.5 \end{bmatrix}[1 \quad 0]\right| \qquad (13)$$

$$|sI - \hat{A}| = \begin{vmatrix} s + 5.5 & -1 \\ 14.5 & s \end{vmatrix} = s^2 + 5.5s + 14.5 = 0 \qquad \underline{(14)}$$

$$s = -2.75 \pm 2.63i \qquad \underline{(15)}$$