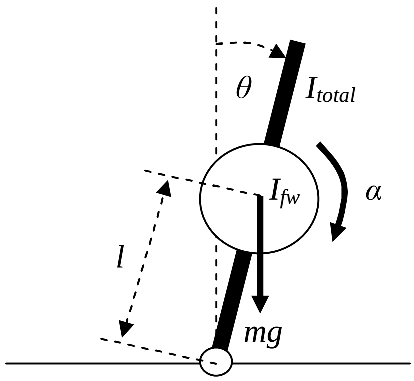
Midterm Examination **Control Theory AT74.02** **October 4, 2011**

Time: 9:00-11:00 hrs. Open Book
Marks: 100

Attempt all questions.

A flywheel is applied to balance an inverted pendulum system as shown in the below figure.



After linearization, the relation between the input of flywheel acceleration, α, and the output of inverted pendulum angle, θ, is represented by the following differential equation.

$$mgl\theta + I_{fw}\alpha = I_{total}\ddot{\theta}$$

When m = 10 kg, g = 10 m/s^2, l = 50 cm, I_{fw} = 20 kg·m^2, I_{total} = 25 kg·m^2.

(a) Determine transfer function, $G(s)$, of this system. What are the roots of the characteristic equation? Is this system asymptotically stable, neutrally stable, or unstable? (10)

Substitute all the parameters.

$$(10\,\text{kg})(10\,\text{m/s}^2)(0.5\,\text{m})\theta + (20\,\text{kg}\cdot\text{m}^2)\alpha = (25\,\text{kg}\cdot\text{m}^2)\ddot{\theta} \quad (1)$$

Take Laplace transformation and set all the initial conditions to zero.

$$50\theta + 20\alpha = 25s^2\theta \quad (2)$$

$$\frac{\theta(s)}{\alpha(s)} = G(s) = \frac{20}{25s^2 - 50} = \frac{0.8}{s^2 - 2} \quad (3)$$

The characteristic equation,

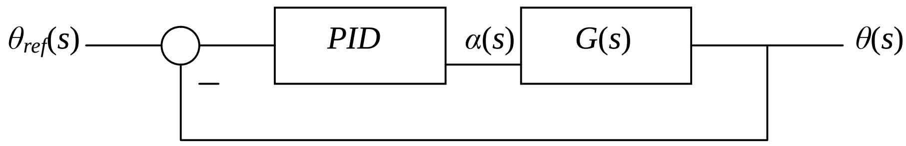
$$s^2 - 2 = 0 \quad (4)$$

Roots of the characteristic equation,

$$s = \pm\sqrt{2} \quad (5)$$

Since there is a positive root, this system is unstable.

(b) If a PID controller is applied to control the system as shown in the below block diagram. Determine the closed loop transfer function and the ranges of the proportional, integral, and derivative gains that make the system become stable. (10)



Closed loop transfer function, $T_c(s)$, is determined.

$$T_c(s) = \frac{0.8K_D s^2 + 0.8K_P s + 0.8K_I}{s^3 + 0.8K_D s^2 + (0.8K_p - 2)s + 0.8K_I} \qquad (6)$$

Characteristic equation,

$$s^3 + 0.8K_D s^2 + (0.8K_p - 2)s + 0.8K_I = 0 \qquad (7)$$

By Routh-Hurwitz table,

	1	$0.8K_P - 2$
α	$0.8K_D$	$0.8K_I$
$1/(0.8K_D)$	$(0.8K_P K_D - 2K_D - K_I)/K_D$	0
$0.8K_D^2/(0.8K_P K_D - 2K_D - K_I)$	$0.8K_I$	0
$(0.8K_P K_D - 2K_D - K_I)/(0.8K_I K_D)$	0	0

The system is asymptotically stable if and only if all α's are positive,

From $1/(0.8K_D)>0$

$$K_D > 0 \qquad (8)$$

From $0.8K_D^2/(0.8K_P K_D - 2K_D - K_I)>0$,

$$0.8K_P K_D - 2K_D - K_I > 0 \qquad (9)$$

$$K_P > 2.5 + 1.25K_I / K_D \qquad (10)$$

From $(0.8K_P K_D - 2K_D - K_I)/(0.8K_I K_D)>0$

$$K_I > 0 \qquad (11)$$

(c) Design the PID controller that makes all the roots of the characteristic equation locate at -2. Determine the steady-state response when the reference input is a unit step function. (20)

Characteristic equation,

$$s^3 + 0.8K_D s^2 + (0.8K_p - 2)s + 0.8K_I = (s+2)^3 = s^3 + 6s^2 + 12s + 8 \qquad (12)$$

Thus,

$$K_D = 7.5 \qquad (13)$$

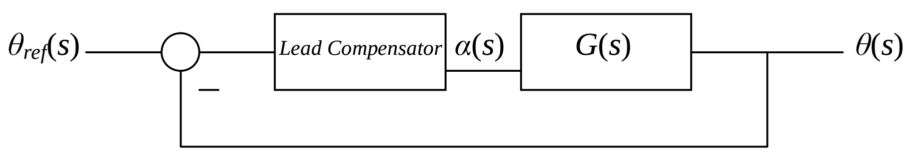
$$K_P = 17.5 \qquad (14)$$

$$K_I = 10 \qquad (15)$$

From Final-Value Theorem,

$$\theta_{ss} = \lim_{s \to 0} s\theta(s) = s \frac{0.8K_D s^2 + 0.8K_P s + 0.8K_I}{s^3 + 0.8K_D s^2 + (0.8K_p - 2)s + 0.8K_I} \frac{1}{s} = 1 \qquad (16)$$

(d) If a lead compensator is applied to control the system as shown in the below block diagram. Determine the closed loop transfer function and the ranges of all the compensator parameters and the gain that make the system become stable. (10)



Closed loop transfer function, $T_c(s)$, is determined.

$$T_c(s) = \frac{0.8Ks + 0.8K / aT}{s^3 + (1/T)s^2 + (0.8K - 2)s + (0.8K/(aT) - 2/T)} \qquad (17)$$

Characteristic equation,

$$s^3 + (1/T)s^2 + (0.8K - 2)s + (0.8K/(aT) - 2/T) = 0 \qquad (18)$$

By Routh-Hurwitz table,

	1	*0.8K -2*
α	*1/T*	*0.8K/(aT)-2/T*
T	*(0.8aK-0.8K)/a*	*0*
a/((0.8aK-0.8K)T)	*0.8K/(aT)-2/T*	*0*
(0.8aK-0.8K)T/(0.8K-2a)	*0*	*0*

The system is asymptotically stable if and only if all α's are positive,

From T>0

$$T > 0 \qquad (19)$$

From a/((0.8aK-0.8K)T)>0

$$a > 1 \qquad (20)$$

From (0.8aK-0.8K)T/(0.8K-2a)>0

$$K > 2.5a \quad (21)$$

(e) Design the lead compensator that makes all the roots of the characteristic equation locate at -2. Determine the steady-state response when the reference input is a unit step function. (20)

Characteristic equation,

$$s^3 + (1/T)s^2 + (0.8K - 2)s + (0.8K/(aT) - 2/T) = (s+2)^3 = s^3 + 6s^2 + 12s + 8 \quad (22)$$

Thus,

$$T = 0.17 \quad (23)$$

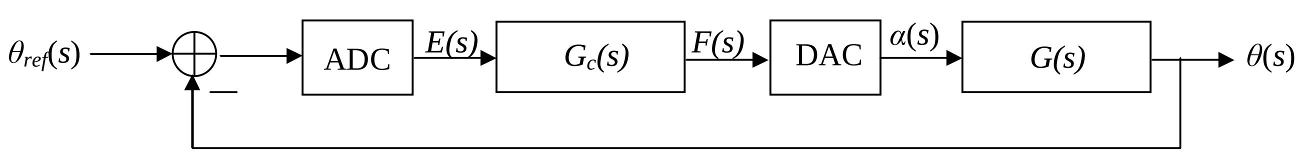
$$K = 17.5 \quad (24)$$

$$a = 4.22 \quad (25)$$

From Final-Value Theorem,

$$\theta_{ss} = \lim_{s \to 0} s\theta(s) = s\frac{0.8Ks + 0.8K/aT}{s^3 + (1/T)s^2 + (0.8K-2)s + (0.8K/(aT) - 2/T)}\frac{1}{s} = \frac{0.8K/aT}{0.8K/(aT) - 2/T} = 2.6 \quad (26)$$

(f) If a digital controller, $G_c(s)$, as shown in the block diagram below is used in the system, design the controller by direct design method when the desired closed loop transfer function, T_c, is represented in s domain by $\frac{1-e^{-0.1s}}{s} \cdot \frac{2}{s+2}$, which is a first order transfer function with zero order hold circuit having unity sensitivity and time constant of 0.5 sec. The sampling time, T, is 0.1 sec. Then determine the control signal at step k, $f(k)$, as a function of control signal at previous steps and error, e, at the current and previous steps. (30)



Firstly determine plant in combination with zero-order hold circuit DAC,

$$G_2(s) = \frac{1}{s} \cdot G(s) = \frac{0.8}{(s^2 - 2)s} = \frac{A}{s} + \frac{B}{s+\sqrt{2}} + \frac{C}{s-\sqrt{2}} = -\frac{0.4}{s} + \frac{0.2}{s+\sqrt{2}} + \frac{0.2}{s-\sqrt{2}} \quad (27)$$

$$G_2(z) = -\frac{0.4z}{(z-1)} + \frac{0.2z}{(z-e^{-0.14})} + \frac{0.2z}{(z-e^{0.14})} \quad (28)$$

$$G = \frac{z-1}{z} G_2(z) = -0.4 + \frac{0.2(z-1)}{(z-e^{-0.14})} + \frac{0.2(z-1)}{(z-e^{0.14})} \tag{29}$$

$$G = \frac{0.2e^{-0.14}z + 0.2e^{0.14}z - 0.4z - 0.4 + 0.2e^{0.14} + 0.2e^{-0.14}}{(z^2 - (e^{-0.14} + e^{0.14})z + 1)} = \frac{0.004z + 0.004}{z^2 - 2.02z + 1} \tag{30}$$

$$T_c(s) = \frac{1-e^{-0.1s}}{s} \cdot \frac{2}{s+2} \tag{31}$$

$$\frac{2}{s(s+2)} = \frac{D}{s} + \frac{E}{s+2} = \frac{1}{s} - \frac{1}{s+2} \Rightarrow \frac{z}{z-1} - \frac{z}{z-e^{-0.2}} \tag{32}$$

$$T_c(z) = 1 - \frac{z-1}{z-e^{-0.2}} = \frac{1-e^{-0.2}}{z-e^{-0.2}} = \frac{0.18}{z-0.82} \tag{33}$$

$$G_D = \frac{T_c}{G(1-T_c)} = \frac{\frac{0.18}{z-0.82}}{\frac{0.004z+0.004}{z^2-2.02z+1}\left(1-\frac{0.18}{z-0.82}\right)} = \frac{45z^2 - 90.9z + 45}{z^2 - 1} \tag{34}$$

$$\frac{f}{e} = \frac{45z^2 - 90.9z + 45}{z^2 - 1} \tag{35}$$

$$(z^2 - 1) f = (45z^2 - 90.9z + 45)e \tag{36}$$

$$\left(1 - \frac{1}{z^2}\right) f = \left(45 - \frac{90.9}{z} + \frac{45}{z^2}\right) e \tag{37}$$

$$f(k) = f(k-2) + 45e(k) - 90.9e(k-1) + 45e(k-2) \tag{38}$$