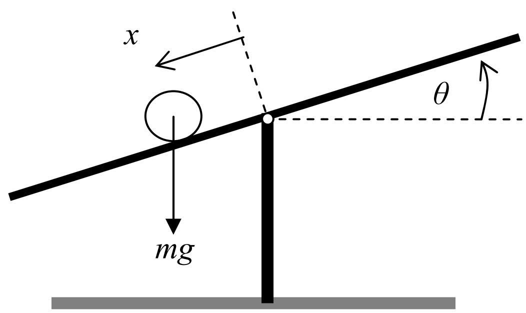
Final Examination **Control Theory AT74.02** **November 21, 2012**

Time: 10:00-12:00 hrs. Open Book

Marks: 100

Attempt all questions.

Consider the system from midterm examination of a ball rolling freely in a rotating rail as shown in the below figure.



After linearization, the relation between the input of rotating angle, θ, and the output of ball position, x, is represented by the following differential equation.

$$g\theta = \ddot{x}$$

When g = 10 m/s^2.

(a) Determine a state-space representation of the system when the state variable x_1 represents the ball position and the state variable x_2 represents the ball velocity. Assume there is a camera used to measure the ball position. (15)

<u>Solution</u>

$$x_1 = x \quad (1)$$

$$x_2 = \dot{x} \quad (2)$$

Thus,

$$\dot{x}_1 = x_2 \quad (3)$$

And

$$\dot{x}_2 = g\theta \quad (4)$$

Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix} [\theta] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} [\theta] \quad \underline{(5)}$$

$$[y] = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underline{(6)}$$

(b) If the ball should be controlled to stop at the position x_r, remodel the state-space system. (5)

Solution

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d \quad (1)$$

When

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x - x_r \\ \dot{x} - 0 \end{bmatrix} \quad (2)$$

$$A - A_r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (3)$$

Thus,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} [\theta] + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_r \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} [\theta] \quad (4)$$

$$[y] = [1 \quad 0] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (5)$$

(c) Design the compensator by separation method. Determine the function of rotating angle that makes the characteristic equation of the regulated system have damping ratio of 0.5 and time constant of 0.25 s. Determine the reduced-order observer gain that makes the observer pole have the same natural frequency as the conjugate poles of the regulator and also determine the equations used to estimate all the state variables. (30)

Solution

Design the regulator,

The required characteristic equation of the regulated system is

$$s^2 + 8s + 64 = 0 \quad (1)$$

$$|sI - A_c| = |sI - A + BG| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} [g_1 \quad g_2] \right| = 0 \quad (2)$$

$$|sI - A_c| = \begin{vmatrix} s & -1 \\ 10g_1 & s + 10g_2 \end{vmatrix} = 0 \quad (3)$$

$$|sI - A_c| = s^2 + 10g_2 s + 10g_1 = s^2 + 8s + 64 \quad (4)$$

Thus,

$$g_1 = 6.4 \quad (5)$$

$$g_2 = 0.8 \quad (6)$$

Thus,

$$[\theta] = -[6.4 \quad 0.8] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (7)$$

Design the observer,

$$C_1 = [1], C_2 = [0], x_1 = [e_1], x_2 = [e_2] \quad (8)$$

$$A_{11} = [0], A_{12} = [1], A_{21} = [0], A_{22} = [0], B_1 = [0], B_2 = [10] \quad (9)$$

$$\hat{x}_1 = e_1 \quad \underline{(10)}$$

$$\hat{x}_2 = Ly + z \quad (11)$$

$$\dot{z} = F\hat{x}_2 + \bar{g}y + Hu \quad (12)$$

$$F = A_{22} - LC_1A_{12} = [0] - [L][1][1] = [-L] \quad (13)$$

$$|sI - F| = [s + L] = [s + 8] \quad (14)$$

$$[L] = [8] \quad (15)$$

$$\bar{g} = (A_{21} - LC_1A_{11})C_1^{-1} = ([0] - [8][1][0])[1]^{-1} = [0] \quad (16)$$

$$H = B_2 - LC_1B_1 = [10] - [8][1][0] = [10] \quad (17)$$

Substitute all the concerned matrices into (11) and (12),

$$\hat{x}_2 = [8]y + z \quad \underline{(18)}$$

$$\dot{z} = [-8]\hat{x}_2 + [10]\theta \quad \underline{(19)}$$

(d) Assume the cost function is expressed by $V = \int_0^\infty (4(x - x_r)^2 + \theta^2)dt$, determine the function of rotating angle that minimizes the cost function. Determine also the characteristic equation and the poles of the regulated system. (25)

Solution

$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \quad (1)$$

$$R = [1] \quad (2)$$

Control signal is determined from

$$\theta = -Ge = -R^{-1}B^t\bar{M}e \quad (3)$$

$$G = [1]^{-1}\begin{bmatrix} 0 \\ 10 \end{bmatrix}^t \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = [1][0 \quad 10]\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \quad (4)$$

$$G = [10m_2 \quad 10m_3] \quad (5)$$

When

$$0 = -\dot{\bar{M}} = \bar{M}A + A^t\bar{M} - \bar{M}BG + Q \quad (6)$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}$$
$$- \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \end{bmatrix}[10m_2 \quad 10m_3] + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \quad (7)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -100m_2^2 + 4 & m_1 - 100m_2m_3 \\ m_1 - 100m_2m_3 & 2m_2 - 100m_3^2 \end{bmatrix} \quad (8)$$

Thus,

$$\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = \begin{bmatrix} 1.26 & 0.2 \\ 0.2 & 0.063 \end{bmatrix} \quad (9)$$

$$G = [2 \quad 0.63] \quad (10)$$

$$[\theta] = -[2 \quad 0.63] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \underline{(11)}$$

Determine characteristic equation of the regulated system,

$$|sI - A_c| = |sI - A + BG| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} [2 \quad 0.63] \right| \quad (12)$$

$$|sI - A_c| = \begin{vmatrix} s & -1 \\ 20 & s + 6.3 \end{vmatrix} = s^2 + 6.3s + 20 = 0 \quad \underline{(13)}$$

$$s = -3.15 \pm 3.17j \quad \underline{(14)}$$

(e) If the ball is disturbed by Gaussian white noise disturbance, v, with power spectral density of the disturbance of 0.2 as expressed by the equation, $g\theta + v = \ddot{x}$, and the output reading of the ball position is contaminated by Gaussian white noise, w, with power spectral density of 0.04, determine Kalman filter gain, characteristic equation of the optimal observer, and its poles. (25)

<u>Solution</u>

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d \quad (1)$$

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} [\theta] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [v] \quad (2)$$

$$[y] = [1 \quad 0] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + [w] \quad (3)$$

$$K = \bar{P} C^t W^{-1} \quad (4)$$

$$K = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} [1 \quad 0]^t [0.04]^{-1} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} [25] = \begin{bmatrix} 25p_1 \\ 25p_2 \end{bmatrix} \quad (5)$$

When

$$0 = \dot{\bar{P}} = A\bar{P} + \bar{P}A^t - KC\bar{P} + FVF^t \quad (6)$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 25p_1 \\ 25p_2 \end{bmatrix} [1 \quad 0] \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0.2][0 \quad 1] \quad (7)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -25p_1^2 + 2p_2 & p_3 - 25p_1p_2 \\ p_3 - 25p_1p_2 & -25p_2^2 + 0.2 \end{bmatrix} \quad (8)$$

Thus,

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0.08 & 0.09 \\ 0.09 & 0.18 \end{bmatrix} \quad (9)$$

$$K = \begin{bmatrix} 2 \\ 2.25 \end{bmatrix} \quad \underline{(10)}$$

Determine characteristic equation of the Kalman filter,

$$|sI - \hat{A}| = |sI - A + KC| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2.25 \end{bmatrix} [1 \quad 0] \right| \quad (11)$$

$$|sI - \hat{A}| = \begin{vmatrix} s + 2 & -1 \\ 2.25 & s \end{vmatrix} = s^2 + 2s + 2.25 = 0 \quad \underline{(12)}$$

$$s = -1 \pm 1.12i \quad \underline{(13)}$$