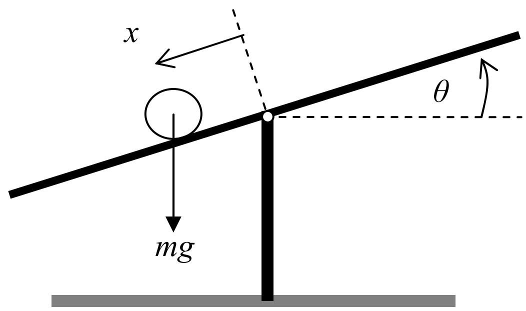
Midterm Examination **Control Theory AT74.02** **October 3, 2012**

Time: 10:00-12:00 hrs. Open Book

Marks: 100

Attempt all questions.

A ball is rolling freely in a rotating rail as shown in the below figure.



After linearization, the relation between the input of rotating angle, θ, and the output of ball position, x, is represented by the following differential equation.

$$g\theta = \ddot{x}$$

When g = 10 m/s^2.

(a) Determine the ball position as a function of time, $x(t)$, when the rotating angle (in radian) is represented by $\theta(t) = 0.1\pi sin(\pi t)$, the initial position and speed follow $x(0) = 0.05$ m, $\dot{x}(0) = -0.1$ m/s. (10)

Substitute all the parameters.

$$(10\text{ m/s}^2)\theta = \ddot{x} \tag{1}$$

Take Laplace transformation and consider all the initial conditions.

$$10\Theta = s^2 X - sx(0) - \dot{x}(0) \tag{2}$$

$$X = \frac{10\Theta}{s^2} + \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} \tag{3}$$

Take Laplace transformation of the input,

$$\Theta = \frac{0.1\pi^2}{s^2 + \pi^2} \tag{4}$$

Substitute the input and all the initial conditions,

$$X = \frac{\pi^2}{s^2\left(s^2+\pi^2\right)} + \frac{0.05}{s} - \frac{0.1}{s^2} = \frac{1}{s^2} - \frac{1}{s^2+\pi^2} + \frac{0.05}{s} - \frac{0.1}{s^2} = \frac{0.9}{s^2} - \frac{1}{s^2+\pi^2} + \frac{0.05}{s} \tag{5}$$

Take the inverse Laplace transformation,

$$x(t) = 0.9t - \frac{1}{\pi}\sin(\pi t) + 0.05 \tag{6}$$

(b) Determine transfer function, $G(s)$, of this system. What are the roots of the characteristic equation? Is this system asymptotically stable, neutrally stable, or unstable? (5)

$$\frac{X}{\Theta} = G(s) = \frac{10}{s^2} \tag{7}$$

The characteristic equation,

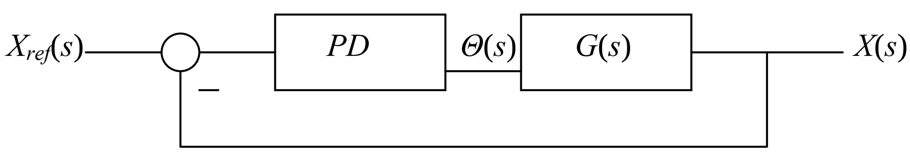
$$s^2 = 0 \tag{8}$$

Roots of the characteristic equation,

$$s = 0,0 \tag{9}$$

Since there are repeated roots at origin point, this system is unstable

(c) If a PD controller is applied to control the system as shown in the below block diagram. Determine the closed loop transfer function and the ranges of the proportional, and derivative gains that make the system become stable. (10)



Closed loop transfer function, $T_c(s)$, is determined.

$$T_c(s) = \frac{10K_D s + 10K_p}{s^2 + 10K_D s + 10K_p} \tag{10}$$

Characteristic equation,

$$s^2 + 10K_D s + 10K_p = 0 \tag{11}$$

By Routh-Hurwitz table,

	1	$10K_P$
α	$10K_D$	0
$1/(10K_D)$	$10K_P$	0
K_D/K_P	0	0

The system is asymptotically stable if and only if all α's are positive, *From $1/(10K_D)>0$*

$$K_D > 0 \tag{12}$$

From $K_D/K_P>0$,

$$K_P > 0 \tag{13}$$

(d) Design the PD controller to have the damping ratio of 0.5 and time constant of 0.25 s. Determine the steady-state response when the reference input is a unit step function. (20)

Characteristic equation,

$$s^2 + 10K_D s + 10K_p = s^2 + 8s + 64 \tag{14}$$

Thus,

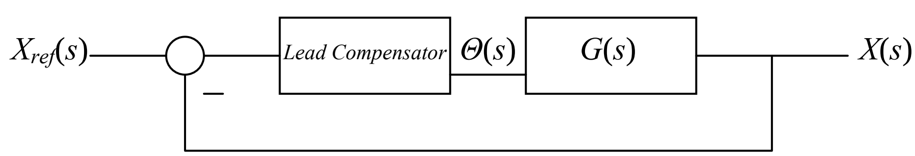
$$K_D = 0.8 \tag{15}$$

$$K_P = 6.4 \tag{16}$$

From Final-Value Theorem,

$$x_{ss} = \lim_{s\to 0} sX(s) = s\frac{10K_D s + 10K_p}{s^2 + 10K_D s + 10K_p}\frac{1}{s} = 1 \tag{17}$$

(e) If a lead compensator is applied to control the system as shown in the below block diagram. Determine the closed loop transfer function and the ranges of all the compensator parameters and the gain that make the system become stable. (10)



Closed loop transfer function, $T_c(s)$, is determined.

$$T_c(s) = \frac{10Ks + 10K/(aT)}{s^3 + (1/T)s^2 + 10Ks + 10K/(aT)} \tag{18}$$

Characteristic equation,

$$s^3 + (1/T)s^2 + 10Ks + 10K/(aT) = 0 \tag{19}$$

By Routh-Hurwitz table,

	1	*10K*
α	*1/T*	*10K/(aT)*

T	*(10aK-10K)/a*	*0*
a/((10aK-10K)T)	*10K/(aT)*	*0*
(a-1)T	*0*	*0*

The system is asymptotically stable if and only if all α's are positive,

From T>0

$$T > 0 \tag{20}$$

From a/((10aK-10K)T)>0

$$K > 0 \tag{21}$$

From (a-1)T >0

$$a > 1 \tag{22}$$

(f) Design the lead compensator so that the conjugate poles have the damping ratio of 0.5 and time constant of 0.25 s and the remaining pole has the same natural frequency as the conjugate poles. Determine the steady-state response when the reference input is a unit step function. (20)

Characteristic equation,

$$s^3 + (1/T)s^2 + 10Ks + 10K/(aT) = (s^2 + 8s + 64)(s+8) = s^3 + 16s^2 + 128s + 512 \tag{23}$$

Thus,

$$T = 0.0625 \tag{24}$$

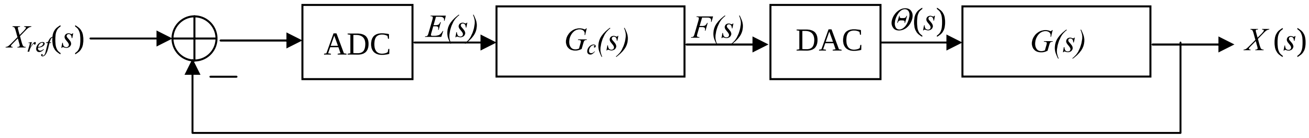
$$K = 12.8 \tag{25}$$

$$a = 4 \tag{26}$$

From Final-Value Theorem,

$$x_{ss} = \lim_{s\to 0} sX(s) = s\frac{10Ks + 10K/(aT)}{s^3 + (1/T)s^2 + 10Ks + 10K/(aT)}\frac{1}{s} = 1 \tag{27}$$

(g) If a digital controller, $G_c(s)$, as shown in the block diagram below is used in the system, design the controller by direct design method when the desired closed loop transfer function, T_c, is represented in s domain by $\frac{1-e^{-0.1s}}{s}\cdot\frac{64}{s^2+8s+64}$, which is a second order transfer function with zero order hold circuit having unity sensitivity, damping ratio of 0.5, and time constant of 0.25 sec. The sampling time, T, is 0.1 sec. Then determine the control signal at step k, $f(k)$, as a function of control signal at previous steps and error, e, at the current and previous steps. (25)



Firstly determine plant in combination with zero-order hold circuit DAC,

$$G_2(s) = \frac{1}{s} \cdot G(s) = \frac{10}{s^3} \tag{28}$$

$$G_2(z) = \frac{z(z+1)0.1}{2(z-1)^3} \tag{29}$$

$$G = \frac{z-1}{z} G_2(z) = \frac{(z+1)0.1}{2(z-1)^2} \tag{30}$$

$$T_c(s) = \frac{1-e^{-0.1s}}{s} \cdot \frac{64}{s^2+8s+64} \tag{31}$$

$$\frac{64}{s(s^2+8s+64)} = \frac{A}{s} + \frac{Bs+C}{s^2+8s+64} = \frac{1}{s} - \frac{s+8}{s^2+8s+64} = \frac{1}{s} - \frac{(s+4)+(4\sqrt{3})/\sqrt{3}}{(s+4)^2+(4\sqrt{3})^2} \tag{32}$$

Take Z transformation,

$$\frac{1}{s} - \frac{(s+4)+(4\sqrt{3})/\sqrt{3}}{(s+4)^2+(4\sqrt{3})^2} \Rightarrow \frac{z}{z-1} - \frac{z^2 - 2e^{-0.4}\cos 0.69 + 0.58ze^{-0.4}\sin 0.69}{z^2 - 2ze^{-0.4}\cos 0.69 + e^{-0.8}} \tag{33}$$

$$T_c(z) = 1 - \frac{(z-1)(z^2 - 2e^{-0.4}\cos 0.69 + 0.58ze^{-0.4}\sin 0.69)}{z(z^2 - 2ze^{-0.4}\cos 0.69 + e^{-0.8})} \tag{34}$$

$$T_c(z) = \frac{z^2(1-2e^{-0.4}\cos 0.69 - 0.58e^{-0.4}\sin 0.69) + z(e^{-0.8} + 2e^{-0.4}\cos 0.69 + 0.58e^{-0.4}\sin 0.69) - 2e^{-0.4}\cos 0.69}{z^3 - 2z^2e^{-0.4}\cos 0.69 + ze^{-0.8}} \tag{35}$$

$$T_c(z) = \frac{-0.28z^2 + 1.73z - 1.03}{z^3 - 1.03z^2 + 0.45z} \tag{36}$$

$$G_D = \frac{T_c}{G(1-T_c)} = \frac{\dfrac{-0.28z^2 + 1.73z - 1.03}{z^3 - 1.03z^2 + 0.45z}}{\dfrac{(z+1)0.1}{2(z-1)^2}\left(1 - \dfrac{-0.28z^2 + 1.73z - 1.03}{z^3 - 1.03z^2 + 0.45z}\right)} \tag{37}$$

$$G_D = \frac{-0.56z^4 + 4.58z^3 - 9.54z^2 + 7.58z - 2.06}{0.100z^4 + 0.025z^3 - 0.203z^2 - 0.025z + 0.103} = \frac{-5.6z^4 + 45.8z^3 - 95.4z^2 + 75.8z - 20.6}{z^4 + 0.25z^3 - 2.03z^2 - 0.25z + 1.03} \tag{38}$$

$$\frac{f}{e} = \frac{-5.6z^4 + 45.8z^3 - 95.4z^2 + 75.8z - 20.6}{z^4 + 0.25z^3 - 2.03z^2 - 0.25z + 1.03} \tag{39}$$

$$(z^4 + 0.25z^3 - 2.03z^2 - 0.25z + 1.03)f = (-5.6z^4 + 45.8z^3 - 95.4z^2 + 75.8z - 20.6)e \tag{40}$$

$$(1 + \frac{0.25}{z} - \frac{2.03}{z^2} - \frac{0.25}{z^3} + \frac{1.03}{z^4})f = (-5.6 + \frac{45.8}{z} - \frac{95.4}{z^2} + \frac{75.8}{z^3} - \frac{20.6}{z^4})e \tag{41}$$

$$f(k) = -0.25f(k-1) + 2.03f(k-2) + 0.25f(k-3) - 1.03f(k-4) - 5.6e(k) + 45.8e(k-1) - 95.4e(k-2) + 75.8e(k-3) - 20.6e(k-4) \tag{42}$$