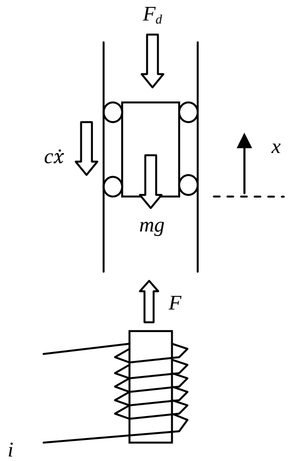
Final Examination **Control Theory AT74.02** **November 19, 2013**

Time: 10:00-12:00 hrs. Open Book
Marks: 100

Attempt all questions.

Consider the slider system from midterm examination again. Repulsive electromagnetic force, F, which linearly varies with the supplied current, i, is used to lift a magnetic slider with mass, m, under the viscous friction with viscosity coefficient, c, and disturbance force, F_d , as shown in the below figure.



The relation between the supplied current, i, and the slider position, x, can be expressed by

$$k_f i - mg - c\dot{x} - F_d = m\ddot{x}$$

when k_f denotes the electromagnetic force constant.

At the equilibrium reference point where $x = 0$, $k_f i_0 = mg$. By defining $i = i_1 + i_0$, where i_1 denotes additional current from the reference point, then the relation is simplified to

$$k_f i_1 - c\dot{x} - F_d = m\ddot{x}$$

From system identification, m = 400 grams, g = 10 m/s^2, c = 20 Ns/m, and k_f = 500 N/A.

(a) Determine a state-space representation of the system when the state variable x_1 represents the slider position and the state variable x_2 represents the slider velocity, the control signal is the additional current, i_1, the disturbance force is F_d . Assume a potentiometer is used to measure the slider position. (15)

<u>Solution</u>

$$x_1 = x \quad (1)$$

$$x_2 = \dot{x} \quad (2)$$

Thus,

$$\dot{x}_1 = x_2 \quad (3)$$

And from

$$k_f i_1 - c\dot{x} - F_d = m\ddot{x} \tag{4}$$

$$500 i_1 - 20\dot{x} - F_d = 0.4\ddot{x} \tag{5}$$

$$\dot{x}_2 = -50x_2 + 1250 i_1 - 2.5F_d \tag{6}$$

Thus,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -50\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1250\end{bmatrix}[i_1] - \begin{bmatrix}0\\ 2.5\end{bmatrix}[F_d] \tag{7}$$

$$[y] = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{8}$$

(b) If the slider should be controlled to stop at the position, x_r, remodel the state-space system by taking into consideration the reference. (5)

<u>Solution</u>

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d \tag{1}$$

When

$$\begin{bmatrix}e_1\\ e_2\end{bmatrix} = \begin{bmatrix}x - x_r\\ \dot{x} - 0\end{bmatrix} \tag{2}$$

$$A - A_r = \begin{bmatrix}0 & 1\\ 0 & -50\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -50\end{bmatrix} \tag{3}$$

Thus,

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -50\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} + \begin{bmatrix}0\\ 1250\end{bmatrix}[i_1] + \begin{bmatrix}0 & 1\\ 0 & -50\end{bmatrix}\begin{bmatrix}x_r\\ 0\end{bmatrix} - \begin{bmatrix}0\\ 2.5\end{bmatrix}[F_d] \tag{4}$$

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -50\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} + \begin{bmatrix}0\\ 1250\end{bmatrix}[i_1] - \begin{bmatrix}0\\ 2.5\end{bmatrix}[F_d] \tag{5}$$

$$[y] = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} \tag{6}$$

(c) Design the compensator by separation method. The regulator has to place the conjugate poles to have 0.5 damping ratio with natural frequency of 100 rad/s and keeps the slider position at the reference position even with disturbance force. Write the additional current as a function of state errors and disturbance force. The reduced-order observer is designed to place the observer poles to have 0.707 damping ratio with time constant of 5 ms when the disturbance force is assumed as a step function. Write the equations that are used to estimate the state errors and the disturbance. (30)

<u>Solution</u>

Design the regulator,

The desired characteristic equation of the regulated system is

$$s^2 + 100s + 10000 = 0 \quad (1)$$

$$|sI - A_c| = |sI - A + BG| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -50 \end{bmatrix} + \begin{bmatrix} 0 \\ 1250 \end{bmatrix} [g_1 \quad g_2] \right| = 0 \quad (2)$$

$$|sI - A_c| = \begin{vmatrix} s & -1 \\ 1250g_1 & s + 50 + 1250g_2 \end{vmatrix} = 0 \quad (3)$$

$$|sI - A_c| = s^2 + 50s + 1250g_2 s + 1250g_1 = s^2 + 100s + 10000 \quad (4)$$

Thus,

$$g_1 = 8 \quad (5)$$

$$g_2 = 0.04 \quad (6)$$

The gain for disturbance force,

$$g_0 = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1}E \quad (7)$$

$$A - BG = \begin{bmatrix} 0 & 1 \\ -10000 & -100 \end{bmatrix} \quad (8)$$

$$[A - BG]^{-1} = \frac{1}{10000} \begin{bmatrix} -100 & -1 \\ 10000 & 0 \end{bmatrix} \quad (9)$$

$$C[A - BG]^{-1} = \frac{1}{10000} [1 \quad 0] \begin{bmatrix} -100 & -1 \\ 10000 & 0 \end{bmatrix} = \frac{1}{10000} [-100 \quad -1] \quad (10)$$

$$C[A - BG]^{-1}B = \frac{1}{10000} [-100 \quad -1] \begin{bmatrix} 0 \\ 1250 \end{bmatrix} = -0.125 \quad (11)$$

$$[C[A - BG]^{-1}B]^{-1} = -8 \quad (12)$$

$$g_0 = (-8) \frac{1}{10000} [-100 \quad -1] \begin{bmatrix} 0 \\ -2.5 \end{bmatrix} = -0.002 \quad (13)$$

Thus,

$$[i_1] = -[8 \quad 0.04] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + [0.002][F_d] \quad (14)$$

Remodel into meta-state form,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{F}_d \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -50 & -2.5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ F_d \end{bmatrix} + \begin{bmatrix} 0 \\ 1250 \\ 0 \end{bmatrix} [i_1] \quad (15)$$

$$[y] = [1 \quad 0 \quad 0] \begin{bmatrix} e_1 \\ e_2 \\ F_d \end{bmatrix} \quad (16)$$

Design the reduced-order observer,

$$C_1 = [1], C_2 = [0 \quad 0], x_1 = [e_1], x_2 = \begin{bmatrix} e_2 \\ F_d \end{bmatrix} \quad (17)$$

$$A_{11} = [0], A_{12} = [1 \quad 0], A_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, A_{22} = \begin{bmatrix} -50 & -2.5 \\ 0 & 0 \end{bmatrix}, B_1 = [0], B_2 = \begin{bmatrix} 1250 \\ 0 \end{bmatrix} \quad (18)$$

$$\hat{x}_1 = e_1 \quad (19)$$

$$\hat{x}_2 = Ly + z \quad (20)$$

$$\dot{z} = F\hat{x}_2 + \bar{g}y + Hu \tag{21}$$

$$F = A_{22} - LC_1A_{12} = \begin{bmatrix} -50 & -2.5 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [1][1 \quad 0] = \begin{bmatrix} -50 - l_1 & -2.5 \\ -l_2 & 0 \end{bmatrix} \tag{22}$$

$$|sI - F| = \begin{vmatrix} s + 50 + l_1 & 2.5 \\ l_2 & s \end{vmatrix} = s^2 + (50 + l_1)s - 2.5l_2 \tag{23}$$

The desired characteristic equation of the reduced-order observer is

$$(s + 200 + 200j)(s + 200 - 200j) = s^2 + 400s + 80000 = 0 \tag{24}$$

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 350 \\ -32000 \end{bmatrix} \tag{25}$$

$$F = \begin{bmatrix} -400 & -2.5 \\ 32000 & 0 \end{bmatrix} \tag{26}$$

$$\bar{g} = (A_{21} - LC_1A_{11})C_1^{-1} = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 350 \\ -32000 \end{bmatrix} [1][0] \right) [1]^{-1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{27}$$

$$H = B_2 - LC_1B_1 = \begin{bmatrix} 1250 \\ 0 \end{bmatrix} - \begin{bmatrix} 350 \\ -32000 \end{bmatrix} [1][0] = \begin{bmatrix} 1250 \\ 0 \end{bmatrix} \tag{28}$$

Substitute all the concerned matrices into (20) and (21),

$$\begin{bmatrix} \widehat{e_2} \\ \widehat{F_d} \end{bmatrix} = \begin{bmatrix} 350 \\ -32000 \end{bmatrix} y + z \tag{29}$$

$$\dot{z} = \begin{bmatrix} -400 & -2.5 \\ 32000 & 0 \end{bmatrix} \begin{bmatrix} \widehat{e_2} \\ \widehat{F_d} \end{bmatrix} + \begin{bmatrix} 1250 \\ 0 \end{bmatrix} [i_1] \tag{30}$$

(d) Assume the cost function is expressed by $V = \int_0^\infty ((x - x_r)^2 + v^2)dt$, when $i_1 = \overline{\iota_1} + v$ and $\overline{\iota_1}$ is the additional current that keeps the slider position at the reference position even with a step disturbance force. Determine the function of additional current as a function of state errors and disturbance force that minimizes the cost function, the characteristic equation, and the optimal regulated poles. (25)

<u>Solution</u>

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

$$R = [1] \tag{2}$$

Control signal is determined from

$$i_1 = -R^{-1}B^t\bar{M}_1e - R^{-1}B^t\bar{M}_2F_d \tag{3}$$

$$R^{-1}B^t\bar{M}_1 = [1]^{-1} \begin{bmatrix} 0 \\ 1250 \end{bmatrix}^t \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = [1][0 \quad 1250] \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \tag{4}$$

$$R^{-1}B^t\bar{M}_1 = [1250m_2 \quad 1250m_3] \tag{5}$$

When

$$0 = -\dot{\bar{M}} = \bar{M}A + A^t\bar{M} - \bar{M}BR^{-1}B^t\bar{M}_1 + Q \tag{6}$$

Substitute all the concerned matrices,

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}m_1 & m_2\\m_2 & m_3\end{bmatrix}\begin{bmatrix}0 & 1\\0 & -50\end{bmatrix} + \begin{bmatrix}0 & 0\\1 & -50\end{bmatrix}\begin{bmatrix}m_1 & m_2\\m_2 & m_3\end{bmatrix}$$
$$-\begin{bmatrix}m_1 & m_2\\m_2 & m_3\end{bmatrix}\begin{bmatrix}0\\1250\end{bmatrix}[1250m_2 \quad 1250m_3] + \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \tag{7}$$

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}-1562500m_2^2 + 1 & m_1 - 50m_2 - 1562500m_2m_3\\m_1 - 50m_2 - 1562500m_2m_3 & 2m_2 - 100m_3 - 1562500m_3^2\end{bmatrix} \tag{8}$$

Thus,

$$\begin{bmatrix}m_1 & m_2\\m_2 & m_3\end{bmatrix} = \begin{bmatrix}0.0566 & 0.0008\\0.0008 & 0.00001325\end{bmatrix} \tag{9}$$

$$R^{-1}B^t\bar{M}_1 = [1 \quad 0.0166] \tag{10}$$

$$\bar{M}_2 = -A_c^{t^{-1}}\bar{M}_1E = -\left(\begin{bmatrix}0 & 1\\0 & -50\end{bmatrix} - \begin{bmatrix}0\\1250\end{bmatrix}[1 \quad 0.0166]\right)^{t^{-1}}\begin{bmatrix}0.0566 & 0.0008\\0.0008 & 0.00001325\end{bmatrix}\begin{bmatrix}0\\-2.5\end{bmatrix} \tag{11}$$

$$\bar{M}_2 = \begin{bmatrix}0.0000801\\0.0000016\end{bmatrix} \tag{12}$$

$$R^{-1}B^t\bar{M}_2 = [1]^{-1}\begin{bmatrix}0\\1250\end{bmatrix}^t\begin{bmatrix}0.0000801\\0.0000016\end{bmatrix} = [1][0 \quad 1250]\begin{bmatrix}0.0000801\\0.0000016\end{bmatrix} \tag{13}$$

$$R^{-1}B^t\bar{M}_2 = [0.002] \tag{14}$$

$$i_1 = -[1 \quad 0.0166]\begin{bmatrix}e_1\\e_2\end{bmatrix} - [0.002][F_d] \tag{15}$$

$$|sI - A_c| = \left|\begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\0 & -50\end{bmatrix} + \begin{bmatrix}0\\1250\end{bmatrix}[1 \quad 0.0166]\right| = \begin{vmatrix}s & -1\\1250 & s + 70.75\end{vmatrix} \tag{16}$$

$$|sI - A_c| = s^2 + 70.75s + 1250 = (s + 35.355)(s + 35.355) = 0 \tag{17}$$

(e) If the disturbance force, F_d, is Gaussian white noise with power spectral density of 2, and the output reading of the slider position is contaminated by Gaussian white noise, *w*, with power spectral density of 0.5, determine Kalman filter gain, characteristic equation of the optimal observer, and its poles. (25)

Solution

$$\begin{bmatrix}\dot{e}_1\\\dot{e}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & -50\end{bmatrix}\begin{bmatrix}e_1\\e_2\end{bmatrix} + \begin{bmatrix}0\\1250\end{bmatrix}[i_1] - \begin{bmatrix}0\\2.5\end{bmatrix}[F_d] \tag{1}$$

$$[y] = [1 \quad 0]\begin{bmatrix}e_1\\e_2\end{bmatrix} \tag{2}$$

$$K = \bar{P}C^tW^{-1} \tag{3}$$

$$K = \begin{bmatrix}p_1 & p_2\\p_2 & p_3\end{bmatrix}[1 \quad 0]^t[0.5]^{-1} = \begin{bmatrix}p_1 & p_2\\p_2 & p_3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}[2] = \begin{bmatrix}2p_1\\2p_2\end{bmatrix} \tag{4}$$

When

$$0 = \dot{\bar{P}} = A\bar{P} + \bar{P}A^t - KC\bar{P} + FVF^t \tag{5}$$

Substitute all the concerned matrices,

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & -50\end{bmatrix}\begin{bmatrix}p_1 & p_2\\p_2 & p_3\end{bmatrix} + \begin{bmatrix}p_1 & p_2\\p_2 & p_3\end{bmatrix}\begin{bmatrix}0 & 0\\1 & -50\end{bmatrix}$$
$$-\begin{bmatrix}2p_1\\2p_2\end{bmatrix}[1 \quad 0]\begin{bmatrix}p_1 & p_2\\p_2 & p_3\end{bmatrix} + \begin{bmatrix}0\\2.5\end{bmatrix}[2][0 \quad 2.5] \tag{6}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2p_1^2 + 2p_2 & p_3 - 50p_2 - 2p_1p_2 \\ p_3 - 50p_2 - 2p_1p_2 & -2p_2^2 - 100p_3 + 12.5 \end{bmatrix} \quad (7)$$

Thus,

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0.05 & 0.0025 \\ 0.0025 & 0.125 \end{bmatrix} \quad (8)$$

$$K = \begin{bmatrix} 0.1 \\ 0.005 \end{bmatrix} \quad (9)$$

Determine characteristic equation of the Kalman filter,

$$\left|sI - \hat{A}\right| = |sI - A + KC| = \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -50 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.005 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}\right| \quad (10)$$

$$\left|sI - \hat{A}\right| = \begin{vmatrix} s + 0.1 & -1 \\ 0.005 & s + 50 \end{vmatrix} = s^2 + 50.1s + 0.005 = (s + 50.1)(s + 0.0001) = 0 \quad (11)$$