Final Examination **Control Theory AT74.02** **November 18, 2014**

Time: 10:00-12:00 hrs. Open Book

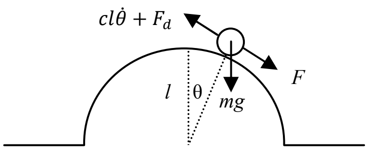
Marks: 100

Attempt all questions.

Consider the ball falling from a circular hill system modified from midterm examination again. A tangential force, F, is used to control angular position, θ, of a ball falling from a circular hill in the environment with viscosity friction,$cl\dot{\theta}$, and aerodynamic disturbance force, F_d, as shown in the below figure. When the ball position is near the hill top, the relation between the tangential force and the ball position can be linearlized as expressed by

$$F - F_d - cl\dot{\theta} + mg\theta = ml\ddot{\theta}$$

When the mass, m, is 5 kg, gravitational acceleration, g, is 10 m/s^2, the hill radius, l, is 2 m, and the viscosity friction coefficient, c, is 20 N·s/(m·rad).



(a) Determine a state-space representation of the system when the state variable x_1 represents the ball angular position and the state variable x_2 represents the ball angular velocity, the control signal is the tangential force, F, the disturbance is the aerodynamics force, F_d. Assume a potentiometer is used to measure the ball angular position. (15)

Solution

$$x_1 = \theta \tag{1}$$

$$x_2 = \dot{\theta} \tag{2}$$

Thus,

$$\dot{x}_1 = x_2 \tag{3}$$

And from

$$F - F_d - clx_2 + mgx_1 = ml\dot{x}_2 \tag{4}$$

$$F - F_d - (20 \times 2)x_2 + (5 \times 10)x_1 = (5 \times 2)\dot{x}_2 \tag{5}$$

$$\dot{x}_2 = 5x_1 - 4x_2 + 0.1F - 0.1F_d \tag{6}$$

Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} [F] + \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} [F_d] \tag{7}$$

$$[y] = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{8}$$

(b) If the ball is controlled to stop at an angular position, θ_r, remodel the state-space system by taking into consideration the reference. (5)

Solution

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d = Ae + Bu + Ex_0 \tag{1}$$

When

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \theta - \theta_r \\ \dot{\theta} - 0 \end{bmatrix} \tag{2}$$

$$A - A_r = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} \tag{3}$$

Thus,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}[F] + \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} \theta_r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -0.1 \end{bmatrix}[F_d] \tag{4}$$

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}[F] + \begin{bmatrix} 0 & 1 & 0 \\ 5 & -4 & -0.1 \end{bmatrix}\begin{bmatrix} \theta_r \\ 0 \\ F_d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}[F] + \begin{bmatrix} 0 & 0 \\ 5 & -0.1 \end{bmatrix}\begin{bmatrix} \theta_r \\ F_d \end{bmatrix} \tag{5}$$

$$[y] = [1 \quad 0]\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{6}$$

(c) Design the compensator by separation method. The regulator has to place the poles to have the time constant of 0.2 second and the damping ratio of $1/\sqrt{2}$. Write the tangential force as a function of the state errors, the references, and the disturbance from aerodynamics force. The reduced-order observer is designed to place all the observer poles at -10 when the disturbance is assumed as a step function. Write the equations that are used to estimate the angular velocity error and the disturbance. The angular position and all the references are either measured or known. (30)

Solution

Design the regulator,

The desired characteristic equation of the regulated system is

$$(s + 5 + 5j)(s + 5 - 5j) = s^2 + 10s + 50 = 0 \tag{1}$$

$$|sI - A_c| = |sI - A + BG| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}[g_1 \quad g_2] \right| = 0 \tag{2}$$

$$|sI - A_c| = \begin{vmatrix} s & -1 \\ -5 + 0.1g_1 & s + 4 + 0.1g_2 \end{vmatrix} = 0 \tag{3}$$

$$|sI - A_c| = s^2 + 4s + 0.1g_2 s - 5 + 0.1g_1 = s^2 + 10s + 50 \tag{4}$$

Thus,

$$g_1 = 550 \tag{5}$$

$$g_2 = 60 \tag{6}$$

The gain for disturbance force,

$$g_0 = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1}E \tag{7}$$

$$A - BG = \begin{bmatrix} 0 & 1 \\ -50 & -10 \end{bmatrix} \tag{8}$$

$$[A - BG]^{-1} = \frac{1}{50}\begin{bmatrix} -10 & -1 \\ 50 & 0 \end{bmatrix} \tag{9}$$

$$C[A - BG]^{-1} = \frac{1}{50}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -10 & -1 \\ 50 & 0 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} -10 & -1 \end{bmatrix} \tag{10}$$

$$C[A - BG]^{-1}B = \frac{1}{50}\begin{bmatrix} -10 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.1 \end{bmatrix} = -0.002 \tag{11}$$

$$[C[A - BG]^{-1}B]^{-1} = -500 \tag{12}$$

$$g_0 = (-500)\frac{1}{50}\begin{bmatrix} -10 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 5 & -0.1 \end{bmatrix} = \begin{bmatrix} 50 & -1 \end{bmatrix} \tag{13}$$

Thus,

$$[F] = -\begin{bmatrix} 550 & 60 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \begin{bmatrix} 50 & -1 \end{bmatrix}\begin{bmatrix} \theta_r \\ F_d \end{bmatrix} \tag{14}$$

Remodel into meta-state form,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{\theta}_r \\ \dot{e}_2 \\ \dot{F}_d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 5 & -4 & -0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ \theta_r \\ e_2 \\ F_d \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}[F] \tag{15}$$

$$[y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ \theta_r \\ e_2 \\ F_d \end{bmatrix} \tag{16}$$

Design the reduced-order observer,

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, x_1 = \begin{bmatrix} e_1 \\ \theta_r \end{bmatrix}, x_2 = \begin{bmatrix} e_2 \\ F_d \end{bmatrix} \tag{17}$$

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_{21} = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} -4 & -0.1 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \tag{18}$$

$$\hat{x}_1 = \begin{bmatrix} e_1 \\ \theta_r \end{bmatrix} \tag{19}$$

$$\hat{x}_2 = Ly + z \tag{20}$$

$$\dot{z} = F\hat{x}_2 + \bar{g}y + Hu \tag{21}$$

$$F = A_{22} - LC_1A_{12} = \begin{bmatrix} -4 & -0.1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -4 - l_{11} & -0.1 \\ -l_{21} & 0 \end{bmatrix} \tag{22}$$

$$|sI - F| = \begin{vmatrix} s + 4 + l_{11} & 0.1 \\ l_{21} & s \end{vmatrix} = s^2 + (4 + l_{11})s - 0.1l_{21} \tag{23}$$

The desired characteristic equation of the reduced-order observer is

$$(s + 10)(s + 10) = s^2 + 20s + 100 = 0 \tag{24}$$

$$\begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ -1000 & 0 \end{bmatrix} \tag{25}$$

$$F = \begin{bmatrix} -20 & -0.1 \\ 1000 & 0 \end{bmatrix} \tag{26}$$

$$\bar{g} = (A_{21} - LC_1A_{11})C_1^{-1} = \left(\begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 16 & 0 \\ -1000 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix} \tag{27}$$

$$H = B_2 - LC_1B_1 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} - \begin{bmatrix} 16 & 0 \\ -1000 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \tag{28}$$

Substitute all the concerned matrices into (20) and (21),

$$\begin{bmatrix} \widehat{e_2} \\ \widehat{F_d} \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ -1000 & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ \theta_r \end{bmatrix} + z \tag{29}$$

$$\dot{z} = \begin{bmatrix} -20 & -0.1 \\ 1000 & 0 \end{bmatrix}\begin{bmatrix} \widehat{e_2} \\ \widehat{F_d} \end{bmatrix} + \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ \theta_r \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}[F] \tag{30}$$

(d) Assume the cost function is expressed by $V = \int_0^\infty ((\theta - \theta_r)^2 + f^2)dt$, when $F = \bar{f} + f$ and $\bar{f}$ is the tangential force that keeps the ball angular position at the reference angular position even with a step aerodynamics disturbance force. Determine the tangential force, F, as a function of the state errors, the references, and the disturbance from aerodynamics force that minimizes the cost function. What are the optimal characteristic equation and its poles? (25)

Solution

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

$$R = [1] \tag{2}$$

Control signal is determined from

$$F = -R^{-1}B^t\bar{M}_1e - R^{-1}B^t\bar{M}_2F_d \tag{3}$$

$$R^{-1}B^t\bar{M}_1 = [1]^{-1}\begin{bmatrix} 0 \\ 0.1 \end{bmatrix}^t\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = [1][0 \quad 0.1]\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \tag{4}$$

$$R^{-1}B^t\bar{M}_1 = [0.1m_2 \quad 0.1m_3] \tag{5}$$

When

$$0 = -\dot{\bar{M}} = \bar{M}A + A^t\bar{M} - \bar{M}BR^{-1}B^t\bar{M}_1 + Q \tag{6}$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}$$
$$- \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}\begin{bmatrix} 0 \\ 0.1 \end{bmatrix}[0.1m_2 \quad 0.1m_3] + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.01m_2^2 + 10m_2 + 1 & m_1 - 4m_2 + 5m_3 - 0.01m_2m_3 \\ m_1 - 4m_2 + 5m_3 - 0.01m_2m_3 & 2m_2 - 8m_3 - 0.01m_3^2 \end{bmatrix} \tag{8}$$

Thus,

$$\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = \begin{bmatrix} 5000.65 & 1000.1 \\ 1000.1 & 200.01 \end{bmatrix} \quad (9)$$

$$R^{-1}B^t\bar{M}_1 = [100.01 \quad 20.00] \quad (10)$$

$$\bar{M}_2 = -A_c^{t^{-1}}\bar{M}_1 E = -\left(\begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} - \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}[100.01 \quad 20.00]\right)^{t^{-1}} \begin{bmatrix} 5000.65 & 1000.1 \\ 1000.1 & 200.01 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 5 & -0.1 \end{bmatrix} \quad (11)$$

$$\bar{M}_2 = \begin{bmatrix} 4999.35 & -99.99 \\ 999.90 & -20 \end{bmatrix} \quad (12)$$

$$R^{-1}B^t\bar{M}_2 = [1]^{-1}\begin{bmatrix} 0 \\ 0.1 \end{bmatrix}^t \begin{bmatrix} 4999.35 & -99.99 \\ 999.90 & -20 \end{bmatrix} = [1][0 \quad 0.1]\begin{bmatrix} 4999.35 & -99.99 \\ 999.90 & -20 \end{bmatrix} \quad (13)$$

$$R^{-1}B^t\bar{M}_2 = [99.99 \quad -2] \quad (14)$$

$$F = -[100.01 \quad 20.00]\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - [99.99 \quad -2]\begin{bmatrix} \theta_r \\ F_d \end{bmatrix} \quad \underline{(15)}$$

$$|sI - A_c| = \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}[100.01 \quad 20.00]\right| = \begin{vmatrix} s & -1 \\ 5 & s+6 \end{vmatrix} \quad (16)$$

$$|sI - A_c| = s^2 + 6s + 5 = (s+5)(s+1) = 0 \quad \underline{(17)}$$

(e) If the aerodynamic disturbance force, F_d, is Gaussian white noise with power spectral density of 5, and the output reading of the ball angular position is contaminated by Gaussian white noise, *w*, with power spectral density of 0.1, determine Kalman filter gain, characteristic equation of the optimal observer, and its poles. (25)

<u>Solution</u>

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}[F] + \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} \theta_r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -0.1 \end{bmatrix}[F_d] \quad (1)$$

$$[y] = [1 \quad 0]\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + [w] \quad (2)$$

$$K = \bar{P}C^tW^{-1} \quad (3)$$

$$K = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}[1 \quad 0]^t[0.1]^{-1} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}[10] = \begin{bmatrix} 10p_1 \\ 10p_2 \end{bmatrix} \quad (4)$$

When

$$0 = \dot{\bar{P}} = A\bar{P} + \bar{P}A^t - KC\bar{P} + FVF^t \quad (5)$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ 1 & -4 \end{bmatrix} - \begin{bmatrix} 10p_1 \\ 10p_2 \end{bmatrix}[1 \quad 0]\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -0.1 \end{bmatrix}[5][0 \quad -0.1] \quad (6)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -10p_1^2 + 2p_2 & 5p_1 + p_3 - 4p_2 - 10p_1p_2 \\ 5p_1 + p_3 - 4p_2 - 10p_1p_2 & -10p_2^2 + 10p_2 - 8p_3 + 0.05 \end{bmatrix} \quad (7)$$

Thus,

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0.2008 & 0.2017 \\ 0.2017 & 0.2075 \end{bmatrix} \quad (8)$$

$$K = \begin{bmatrix} 2.01 \\ 2.02 \end{bmatrix} \quad \underline{(9)}$$

Determine characteristic equation of the Kalman filter,

$$\left|sI - \hat{A}\right| = |sI - A + KC| = \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} + \begin{bmatrix} 2.01 \\ 2.02 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}\right| \tag{10}$$

$$\left|sI - \hat{A}\right| = \begin{vmatrix} s + 2.01 & -1 \\ -2.98 & s + 4 \end{vmatrix} = s^2 + 6.01s + 5.06 = (s + 5.00)(s + 1.01) = 0 \tag{11}$$