Midterm Examination **Control Theory AT74.02** **October 7, 2014**

Time: 10:00-12:00 hrs. Open Book

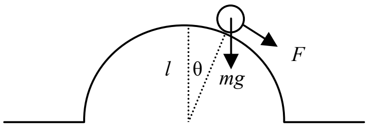
Marks: 100

Attempt all questions.

A tangential force, *F*, is used to control angular position, θ, of a ball falling from a circular hill in viscosity friction environment as shown in the below figure. When the ball position is near the hill top, the relation between the tangential force and the ball position can be linearlized as expressed by

$$F - cl\dot{\theta} + mg\theta = ml\ddot{\theta}$$

When the mass, m, is 5 kg, gravitational acceleration, *g*, is 10 m/s^2, the hill radius, *l*, is 2 m, and the viscosity friction coefficient, *c*, is 20 N·s/(m·rad).



(a) Determine transfer function, *G*(*s*), from tangential force, F, to ball angular position, θ, of this system. What are the roots of the characteristic equation? Is this system asymptotically stable, neutrally stable, or unstable? (10)

Solution

$$F - cl\dot{\theta} + mg\theta = ml\ddot{\theta} \tag{1}$$

By taking Laplace transformation and neglect all the initial conditions,

$$F = (mls^2 + cls - mg)\Theta \tag{2}$$

$$G = \frac{\Theta}{F} = \frac{1}{mls^2+cls-mg} = \frac{1}{5\times2s^2+20\times2s-5\times10} = \frac{1}{10s^2+40s-50} = \frac{1}{10(s+5)(s-1)} \tag{3}$$

Roots of the characteristic equation,

$$s = -5,1 \tag{4}$$

The system is unstable.

(b) Determine the ball angular position as a function of time, $\theta(t)$, when the ball has initial angular velocity of 0.1 rad/sec at the hill top while a constant tangential force of -20 N is applied. (10)

Solution

$$F - cl\dot{\theta} + mg\theta = ml\ddot{\theta} \tag{5}$$

By taking Laplace transformation and consider all the initial conditions,

$$F - cl(s\Theta - \theta(0)) + mg\Theta = ml(s^2\Theta - s\theta(0) - \dot{\theta}(0)) \quad (6)$$

$$F = (mls^2 + cls - mg)\Theta - (mls + cl)\theta(0) - ml\dot{\theta}(0) \quad (7)$$

$$\Theta = \frac{F + (mls + cl)\theta(0) + ml\dot{\theta}(0)}{mls^2 + cls - mg} \quad (8)$$

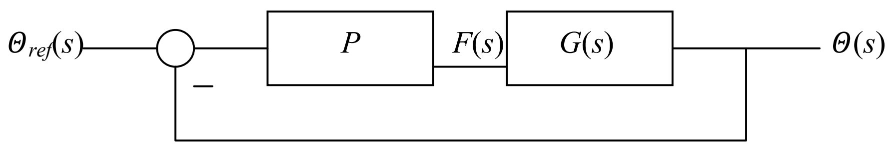
Substitute the step force and initial conditions,

$$\Theta = \frac{-20/s + 5\times 2(0.1)}{5\times 2s^2 + 20\times 2s - 5\times 10} = \frac{-20/s + 1}{10s^2 + 40s - 50} = -\frac{20}{10s(s+5)(s-1)} + \frac{1}{10(s+5)(s-1)} \quad (9)$$

$$\Theta = -\frac{0.067}{s+5} - \frac{0.333}{s-1} + \frac{0.4}{s} - \frac{0.017}{s+5} + \frac{0.017}{s-1} = -\frac{0.084}{s+5} - \frac{0.316}{s-1} + \frac{0.4}{s} \quad (10)$$

$$\theta = -0.084e^{-5t} - 0.316e^{t} + 0.4 \quad \underline{(11)}$$

(c) If a P controller is applied to control the ball position as shown in the below block diagram. Plot the root-locus diagram, Nyquist diagram, and Bode diagram of the system in order to check system stability. In the root-locus diagram determine open-loop zero, pole, slopes of asymptotic lines and their intersection point, break-away/in point and its gain, cross over point and its gain. In the Nyquist diagram, determine points O, A and direction that the Nyquist diagram approaches origin point. In the Bode diagram, determine all the zero, pole, and dc gain. Roughly plot Bode diagram of each zero, pole, and dc gain of the open loop transfer function. (30)



$$G = \frac{\Theta}{F} = \frac{1}{10s^2 + 40s - 50} = \frac{1}{10(s+5)(s-1)} \quad (12)$$

Root locus diagram,

$$\textit{Zero} = \emptyset \quad (13)$$

$$\textit{Pole} = -5, 1 \quad (14)$$

$$e = 2 \quad (15)$$

$$\measuredangle = \pm 90° \quad (16)$$

$$\textit{Intersection point} = -2 \quad (17)$$

Closed loop transfer function, $T_c(s)$, is determined.

$$T_c(s) = \frac{K}{10s^2 + 40s - 50 + K} \quad (18)$$

Break-away point is considered from characteristic polynomial.

$$P = 10s^2 + 40s - 50 + K \quad (19)$$

$$\acute{P} = 20s + 40 = 0 \quad (20)$$

$$s = -2 \quad (21)$$

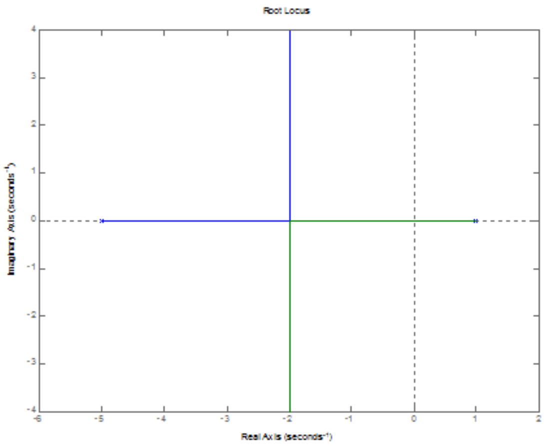
$$0 = 10(-2)^2 + 40(-2) - 50 + K \tag{22}$$

$$K = 90 \tag{23}$$

Cross-over point is obtained by substitution of s with jω in the characteristic equation.

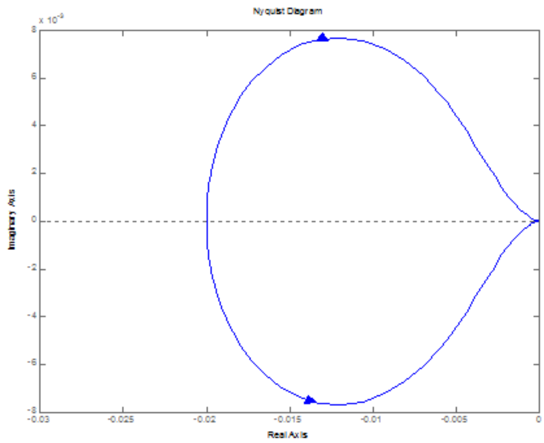
$$0 = -10\omega^2 + 40\omega j - 50 + K \tag{24}$$

$$\omega = 0, K = 50 \tag{25}$$



Nyquist diagram

Point	*s*	*G(s)*
O	*0,+0j*	$-\frac{1}{50}$
A	*ωj*	$\frac{1}{-(10\omega^2+50)+40\omega j} = \frac{-(10\omega^2+50)-40\omega j}{(10\omega^2+50)^2+(40\omega)^2}$
	∞j	$-0-0j$

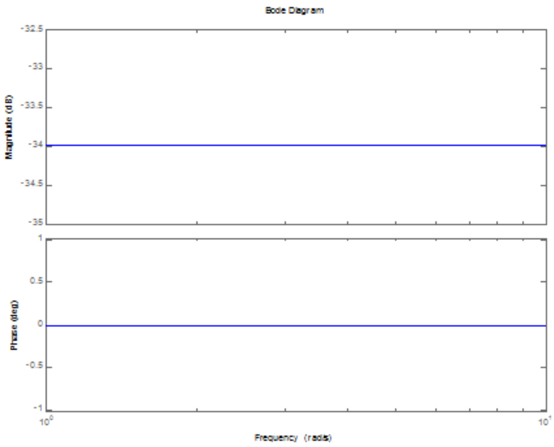


Outside area is unstable, Inside area is stable.

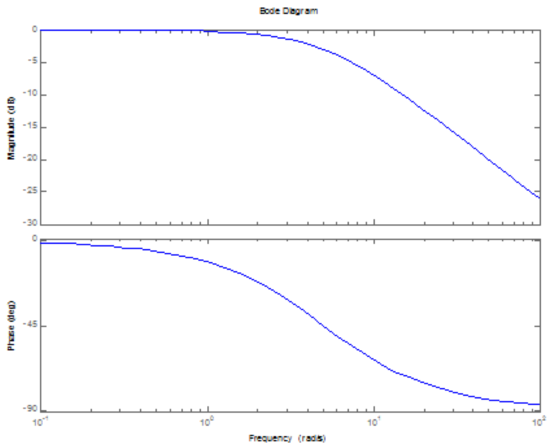
Bode diagram,

$$G = \frac{1}{10(s+5)(s-1)} = \frac{1}{50(s/5+1)(s-1)} = \frac{G_1}{G_2 G_3} \tag{26}$$

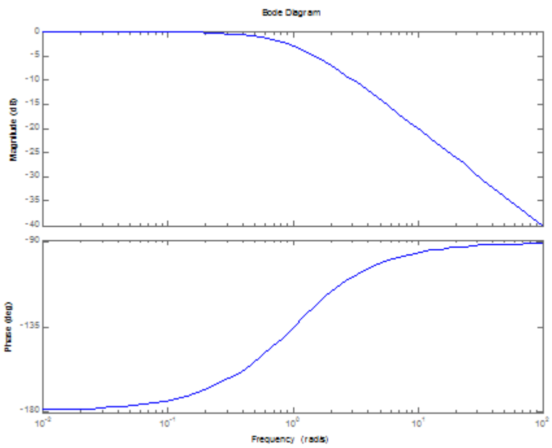
$$G_1 = \frac{1}{50} \tag{27}$$



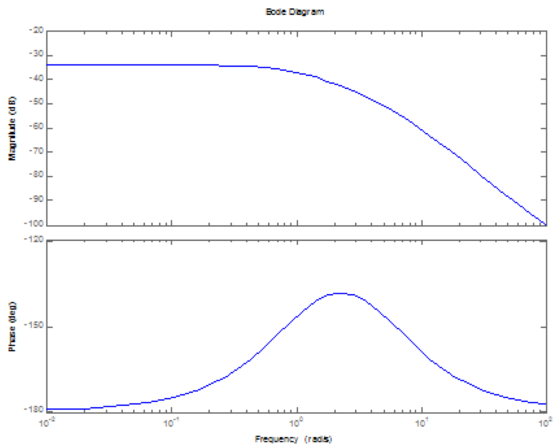
$$G_2 = \frac{1}{(s/5+1)} \tag{28}$$



$$G_3 = \frac{1}{s-1} \tag{29}$$



$$G = \frac{1}{50(s/5+1)(s-1)} = \frac{G_1}{G_2 G_3} \qquad (30)$$



(d) If a PID controller is applied to control the ball position as shown in the below block diagram. Design all the gains that make all the time constants become 0.2 second and the damping ratio of the conjugate roots become $1/\sqrt{2}$. (15)

$\Theta_{ref}(s)$ PID $F(s)$ $G(s)$ $\Theta(s)$

Closed loop transfer function, $T_c(s)$, is determined.

$$T_c(s) = \frac{K_D s^2 + K_P s + K_I}{10s^3 + (K_D+40)s^2 + (K_P-50)s + K_I} \qquad (31)$$

Characteristic equation,

$$10s^3 + (K_D + 40)s^2 + (K_P - 50)s + K_I = s^3 + \left(\frac{K_D}{10} + 4\right)s^2 + \left(\frac{K_P}{10} - 5\right)s + \frac{K_I}{10} = 0 \qquad (32)$$

The desired characteristic equation,

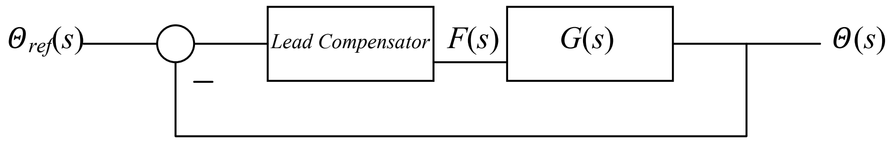
$$(s+5)(s+5+5j)(s+5-5j) = s^3 + 15s^2 + 100s + 250 = 0 \qquad (33)$$

$$K_I = 2500 \qquad (34)$$

$$K_P = 1050 \qquad (35)$$

$$K_D = 110 \qquad (36)$$

(e) If a lead compensator replaces PID controller to control the ball position as shown in the below block diagram. Design all the parameters of the lead compensator and its gain that achieve the same characteristic equation as PID controller has achieved in question (d). Then determine the steady-state response to a step reference ball position at 1 rad. (15)



Closed loop transfer function, $T_c(s)$, is determined.

$$T_c(s) = \frac{Ks+\frac{K}{aT}}{10s^3+\left(\frac{10}{T}+40\right)s^2+(K+\frac{40}{T}-50)s+\frac{K}{aT}-\frac{50}{T}} \quad (37)$$

Characteristic equation,

$$10s^3+\left(\frac{10}{T}+40\right)s^2+(K+\frac{40}{T}-50)s+\frac{K}{aT}-\frac{50}{T} = s^3+\left(\frac{1}{T}+4\right)s^2+(\frac{K}{10}+\frac{4}{T}-5)s+\frac{K}{10aT}-\frac{5}{T}=0 \quad (38)$$

The desired characteristic equation,

$$(s+5)(s+5+5j)(s+5-5j) = s^3+15s^2+100s+250=0 \quad (39)$$

$$T = \frac{1}{11} = 0.091 \quad (40)$$

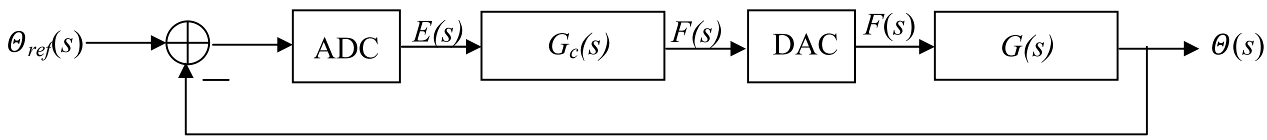
$$K = 610 \quad (41)$$

$$a = 2.2 \quad (42)$$

Steady-state response is determined from final-value theorem.

$$\theta_{ss} = \lim_{s\to 0}\left(s\cdot\frac{Ks+\frac{K}{aT}}{10s^3+\left(\frac{10}{T}+40\right)s^2+(K+\frac{40}{T}-50)s+\frac{K}{aT}-\frac{50}{T}}\cdot\frac{1}{s}\right) = \frac{\frac{K}{aT}}{\frac{K}{aT}-\frac{50}{T}} = 1.22 \quad (43)$$

(f) If a digital controller, $G_c(s)$, as shown in the block diagram below is used in the system, design the controller by direct design method when the desired closed loop transfer function, T_c, is represented in s domain by $\frac{1}{z}$. The sampling time, T, is 0.5 second. Then determine the control signal at step k, $f(k)$, as a function of control signal and error, e, at the current and previous steps. (20)



Firstly determine plant in combination with zero-order hold circuit DAC,

$$G_2(s)=\frac{1}{s}\cdot G(s)=\frac{1}{10s(s+5)(s-1)}=-\frac{0.02}{s}+\frac{0.003}{s+5}+\frac{0.017}{s-1} \tag{44}$$

$$G_2(z)=-\frac{0.02z}{z-1}+\frac{0.003z}{z-0.082}+\frac{0.017z}{z-1.649} \tag{45}$$

$$G=\frac{z-1}{z}G_2(z)=-0.02+\frac{0.003(z-1)}{z-0.082}+\frac{0.017(z-1)}{z-1.649}=\frac{0.0083z+0.0036}{(z-0.082)(z-1.649)} \tag{46}$$

$$T_c(z)=\frac{1}{z} \tag{47}$$

$$G_D=\frac{T_c}{G(1-T_c)}=\frac{\frac{1}{z}}{\frac{0.0083z+0.0036}{(z-0.082)(z-1.649)}\left(1-\frac{1}{z}\right)} \tag{48}$$

$$G_D=\frac{f}{e}=\frac{z^2-1.731z+0.135}{0.0083z^2-0.0047z-0.0036}=\frac{120.482z^2-208.554z+16.265}{z^2-0.566z-0.434} \tag{49}$$

$$(z^2-0.566z-0.434)f=(120.482z^2-208.554z+16.265)e \tag{50}$$

$$(1-\frac{0.566}{z}-\frac{0.434}{z^2})f=(120.482-\frac{208.554}{z}+\frac{16.265}{z^2})e \tag{51}$$

$$f(k)=0.566f(k-1)+0.434f(k-2)+120.482e(k)-208.554e(k-1)+16.265e(k-2) \tag{52}$$