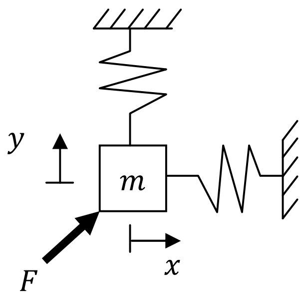
Final Examination **Control Theory AT74.02** **November 17, 2015**

Time: 10:00-12:00 hrs. Open Book

Marks: 100

Attempt all questions.

Consider the 1-mass-2-spring system from midterm examination again. Force, F, is applied to control the coordinate of a mass, m, connected with 2 springs along x and y axes as shown in the below figure. Assume the springs are very long compared with the deflections, thus the springs only provide forces along their axes.



The relation between force and deflection in each axis is expressed by

$$F_x - k_x x = m\ddot{x}$$

$$F_y - k_y y = m\ddot{y}$$

when

$$F = \sqrt{F_x^2 + F_y^2}$$

When the mass, m, is 8 kg, the spring stiffness along x axis, k_x, is 4000 N/m and the spring stiffness along y axis, k_y, is 6400 N/m.

(a) Determine a state-space representation of the system when the state variable x_1 represents the mass position along x axis, the state variable x_2 represents the mass velocity along x axis, x_3 represents the mass position along y axis, the state variable x_4 represents the mass velocity along y axis, the control signals are the force along x axis, F_x, and the force along y axis, F_y. Assume a camera is used to measure the mass positions along x and y axes. (20)

Solution

$$x_1 = x \tag{1}$$

$$x_2 = \dot{x} \tag{2}$$

Thus,

$$\dot{x}_1 = x_2 \tag{3}$$

And from

$$F_x - k_x x = m\ddot{x} \tag{4}$$

$$F_x - 4000x_1 = 8\dot{x}_2 \quad (5)$$

$$\dot{x}_2 = -500x_1 + 0.125F_x \quad (6)$$

$$x_3 = y \quad (7)$$

$$x_4 = \dot{y} \quad (8)$$

Thus,

$$\dot{x}_3 = x_4 \quad (9)$$

And from

$$F_y - k_y y = m\ddot{y} \quad (10)$$

$$F_y - 6400x_3 = 8\dot{x}_4 \quad (11)$$

$$\dot{x}_4 = -800x_3 + 0.125F_y \quad (12)$$

Thus,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -500 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -800 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0.125 & 0\\ 0 & 0\\ 0 & 0.125\end{bmatrix}\begin{bmatrix}F_x\\ F_y\end{bmatrix} \quad \underline{(13)}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \quad \underline{(14)}$$

(b) If the mass is controlled to stop at a coordinate,(x_r, y_r), remodel the state-space system by taking into consideration the references. (20)

Solution

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d = Ae + Bu + Ex_0 \quad (1)$$

When

$$\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\end{bmatrix} = \begin{bmatrix}x - x_r\\ \dot{x} - 0\\ y - y_r\\ \dot{y} - 0\end{bmatrix} \quad (2)$$

$$A - A_r = \begin{bmatrix}0 & 1 & 0 & 0\\ -500 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -800 & 0\end{bmatrix} - \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -500 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -800 & 0\end{bmatrix} \quad (3)$$

Thus,

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{e}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -500 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -800 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0.125 & 0\\ 0 & 0\\ 0 & 0.125\end{bmatrix}\begin{bmatrix}F_x\\ F_y\end{bmatrix} + \begin{bmatrix}0 & 1 & 0 & 0\\ -500 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -800 & 0\end{bmatrix}\begin{bmatrix}x_r\\ 0\\ y_r\\ 0\end{bmatrix} \quad (4)$$

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{e}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ -500 & 0 & 0 & 0\\ 0\ \ 0 & & 0 & 1\\ 0\ \ 0 & & -800 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0.125 & 0\\ 0 & 0\\ 0 & 0.125\end{bmatrix}\begin{bmatrix}F_x\\ F_y\end{bmatrix}+\begin{bmatrix}0 & 0\\ -500 & 0\\ 0 & 0\\ 0 & -800\end{bmatrix}\begin{bmatrix}x_r\\ y_r\end{bmatrix} \quad \underline{(5)}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\end{bmatrix} \quad \underline{(6)}$$

(c) If the force which is used to shape the dynamic only (not to regulate the exogenous input) along x axis is expressed by $F_x = -100(x - x_r) - 100(\dot{x} - \dot{x}_r)$ and the force which is used to shape the dynamic only (not to regulate the exogenous input) along y axis is expressed by $F_y = -200(y - y_r) - 200(\dot{y} - \dot{y}_r)$. Determine characteristic equation of the regulated system and the additional forces along x and y axes that make position error along x and y axes become 0. Then rewrite the total forces along x and y axes that shape the dynamic and handle the exogenous input from the references of the desired mass coordinate. (30)

<u>Solution</u>

Design the regulator,

$|sI - A_c| = |sI - A + BG|$

$$= \left|\begin{bmatrix}s & 0 & 0 & 0\\ 0 & s & 0 & 0\\ 0 & 0 & s & 0\\ 0 & 0 & 0 & s\end{bmatrix}-\begin{bmatrix}0 & 1 & 0 & 0\\ -500 & 0 & 0 & 0\\ 0\ \ 0 & & 0 & 1\\ 0\ \ 0 & & -800 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0.125 & 0\\ 0 & 0\\ 0 & 0.125\end{bmatrix}\begin{bmatrix}g_{11} & g_{12} & 0 & 0\\ 0 & 0 & g_{23} & g_{24}\end{bmatrix}\right|$$

$$= \begin{vmatrix}s & -1 & 0\ \ 0 & \\ 500+0.125g_{11} & s+0.125g_{12} & 0\ \ 0 & \\ 0\ \ 0 & & s & -1\\ 0\ \ 0 & & 800+0.125g_{23} & s+0.125g_{24}\end{vmatrix} = 0 \quad (1)$$

$$|sI - A_c| = s^4 + (0.125g_{12} + 0.125g_{24})s^3 + (0.0156g_{12}g_{24} + 0.125g_{11} + 0.125g_{23} + 1300)s^2$$
$$+(100g_{12} + 62.5g_{24} + 0.0156g_{12}g_{23} + 0.0156g_{11}g_{24})s$$
$$+400000 + 62.5g_{23} + 100g_{11} + 0.0156g_{11}g_{23} = 0 \quad (2)$$

$$\begin{bmatrix}g_{11} & g_{12} & 0 & 0\\ 0 & 0 & g_{23} & g_{24}\end{bmatrix}=\begin{bmatrix}100 & 100 & 0 & 0\\ 0 & 0 & 200 & 200\end{bmatrix} \quad (3)$$

$$|sI - A_c| = s^4 + 37.5s^3 + 1649.5s^2 + 23124s + 422812 = 0 \quad \underline{(4)}$$

The gain for the reference,

$$g_0 = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1}E \quad (5)$$

$$A - BG = \begin{bmatrix}0 & 1 & 0\ \ 0 & \\ -500-0.125g_{11} & -0.125g_{12} & 0\ \ 0 & \\ 0\ \ 0 & & 0 & 1\\ 0\ \ 0 & & -800-0.125g_{23} & -0.125g_{24}\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -512.5 & -12.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -825 & -25 \end{bmatrix} \tag{6}$$

$$[A - BG]^{-1} = \begin{bmatrix} -0.0244 & -0.0020 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -0.0303 & -0.0012 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{7}$$

$$C[A - BG]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -0.0244 & -0.0020 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -0.0303 & -0.0012 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0244 & -0.0020 & 0 & 0 \\ 0 & 0 & -0.0303 & -0.0012 \end{bmatrix} \tag{8}$$

$$C[A - BG]^{-1}B = \begin{bmatrix} -0.0244 & -0.0020 & 0 & 0 \\ 0 & 0 & -0.0303 & -0.0012 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0.125 & 0 \\ 0 & 0 \\ 0 & 0.125 \end{bmatrix} = \begin{bmatrix} -0.00025 & 0 \\ 0 & -0.00015 \end{bmatrix} \tag{9}$$

$$[C[A - BG]^{-1}B]^{-1} = \begin{bmatrix} -4000 & 0 \\ 0 & -6666.67 \end{bmatrix} \tag{10}$$

$$g_0 = [C[A - BG]^{-1}B]^{-1}C[A - BG]^{-1}E$$

$$= \begin{bmatrix} -4000 & 0 \\ 0 & -6666.67 \end{bmatrix} \begin{bmatrix} -0.0244 & -0.0020 & 0 & 0 \\ 0 & 0 & -0.0303 & -0.0012 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -500 & 0 \\ 0 & 0 \\ 0 & -800 \end{bmatrix} = \begin{bmatrix} -4000 & 0 \\ 0 & -6400 \end{bmatrix} \tag{11}$$

Thus,

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = -\begin{bmatrix} 100 & 100 \\ 200 & 200 \end{bmatrix} \begin{bmatrix} x - x_r \\ y - y_r \end{bmatrix} + \begin{bmatrix} 4000 & 0 \\ 0 & 6400 \end{bmatrix} \begin{bmatrix} x_r \\ y_r \end{bmatrix} \tag{12}$$

(d) The reduced-order observer is designed to place all the observer poles at -20 when the references are step functions of the desired positions of the mass. Write the equations that are used to estimate the mass velocities along x and y axes. The mass positions and all the references are either measured or known. Since there are infinite number solutions, use the solution that the observer gain along x axis is a function of position error along x axis only and the observer gain along y axis is a function of position error along y axis only. (30)

Solution

Remodel into meta-state form,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \\ \dot{x}_r \\ \dot{y}_r \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -500 & 0 & 0 & 0 & -500 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -800 & 0 & 0 & -800 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ x_r \\ y_r \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.125 & 0 \\ 0 & 0 \\ 0 & 0.125 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ x_r \\ y_r \end{bmatrix} \tag{2}$$

Design the reduced-order observer,

$$x_1 = \begin{bmatrix} e_1 \\ e_3 \\ x_r \\ y_r \end{bmatrix}, x_2 = \begin{bmatrix} e_2 \\ e_4 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, A_{21} = \begin{bmatrix} -500 & 0 & -500 & 0 \\ 0 & -800 & 0 & -800 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0.125 & 0 \\ 0 & 0.125 \end{bmatrix} \tag{3}$$

$$\hat{x}_1 = \begin{bmatrix} e_1 \\ e_3 \\ x_r \\ y_r \end{bmatrix} \tag{4}$$

$$\hat{x}_2 = Ly + z \tag{5}$$

$$\dot{z} = F\hat{x}_2 + \bar{g}y + Hu \tag{6}$$

From the reduced order observer gains requirement,

$$\begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ 0 & l_{22} & 0 & 0 \end{bmatrix} \tag{7}$$

$$F = A_{22} - LC_1A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ 0 & l_{22} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -l_{11} & 0 \\ 0 & -l_{22} \end{bmatrix} \tag{8}$$

$$|sI - F| = \begin{vmatrix} s + l_{11} & l_{12} \\ l_{21} & s + l_{22} \end{vmatrix} = s^2 + (l_{11} + l_{22})s + l_{11}l_{22} \tag{9}$$

The desired characteristic equation of the reduced-order observer is

$$(s + 20)(s + 20) = s^2 + 40s + 400 = 0 \tag{10}$$

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ 0 & l_{22} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \end{bmatrix} \tag{11}$$

$$F = \begin{bmatrix} -20 & 0 \\ 0 & -20 \end{bmatrix} \tag{12}$$

$\bar{g} = (A_{21} - LC_1A_{11})C_1^{-1}$

$$= \left(\begin{bmatrix} -500 & 0 & -500 & 0 \\ 0 & -800 & 0 & -800 \end{bmatrix} - \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -500 & 0 & -500 & 0 \\ 0 & -800 & 0 & -800 \end{bmatrix} \tag{13}$$

$$H = B_2 - LC_1B_1 = \begin{bmatrix} 0.125 & 0 \\ 0 & 0.125 \end{bmatrix} - \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.125 & 0 \\ 0 & 0.125 \end{bmatrix} \tag{14}$$

Substitute all the concerned matrices into (5) and (6),

$$\begin{bmatrix} \hat{e}_2 \\ \hat{e}_4 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + z \tag{14}$$

$$\dot{z} = \begin{bmatrix} -20 & 0 \\ 0 & -20 \end{bmatrix} \begin{bmatrix} \hat{e}_2 \\ \hat{e}_4 \end{bmatrix} + \begin{bmatrix} -500 & 0 & -500 & 0 \\ 0 & -800 & 0 & -800 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0.125 & 0 \\ 0 & 0.125 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \tag{15}$$