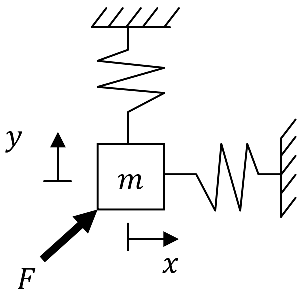
Midterm Examination **Control Theory AT74.02** **October 6, 2015**

Time: 10:00-12:00 hrs. Open Book
Marks: 100

Attempt all questions.

Force, F, is applied to control the coordinate of a mass, m, connected with 2 springs along x and y axes as shown in the below figure. Assume the springs are very long compared with the deflections, thus the springs only provide forces along their axes.



The relation between force and deflection in each axis is expressed by

$$F_x - k_x x = m\ddot{x}$$
$$F_y - k_y y = m\ddot{y}$$

when

$$F = \sqrt{F_x^2 + F_y^2}$$

When the mass, m, is 8 kg, the spring stiffness along x axis, k_x, is 4000 N/m and the spring stiffness along y axis, k_y, is 6400 N/m.

(a) Determine transfer functions, $G_x(s)$ and $G_y(s)$, from force, F_x and F_y, to deflection, x and y, in each axis of this system. (20)

Solution

Along x axis,

$$F_x - k_x x = m\ddot{x} \tag{1}$$

By taking Laplace transformation and neglect all the initial conditions,

$$F_x = (ms^2 + k_x)X \tag{2}$$

$$G_x = \frac{X}{F_x} = \frac{1}{ms^2+k_x} = \frac{1}{8s^2+4000} = \frac{1}{8(s^2+500)} = \frac{1}{8(s+j10\sqrt{5})(s-j10\sqrt{5})} \tag{3}$$

Along y axis,

$$F_y - k_y y = m\ddot{y} \tag{4}$$

By taking Laplace transformation and neglect all the initial conditions,

$$F_y = (ms^2 + k_y)Y \quad (5)$$

$$G_y = \frac{Y}{F_y} = \frac{1}{ms^2+k_y} = \frac{1}{8s^2+6400} = \frac{1}{8(s^2+800)} = \frac{1}{8(s+j20\sqrt{2})(s-j20\sqrt{2})} \quad \underline{(6)}$$

(b) Determine the mass coordinate as a function of time, $(x(t), y(t))$, when the mass is released from the initial coordinate of $(0.05, -0.1)$. What are the oscillation frequencies along x and y axes in Hz. (20)

<u>Solution</u>

Along x axis,

$$F_x - k_x x = m\ddot{x} \quad (7)$$

By taking Laplace transformation and consider all the initial conditions,

$$0 = (ms^2 + k_x)X - msx(0) - m\dot{x}(0) \quad (8)$$

$$X = \frac{msx(0)+m\dot{x}(0)}{ms^2+k_x} = \frac{0.4s}{8s^2+4000} = \frac{0.05s}{s^2+500} \quad (9)$$

$$x(t) = 0.05cos(10\sqrt{5}t) \quad (10)$$

Along y axis,

$$F_y - k_y y = m\ddot{y} \quad (11)$$

By taking Laplace transformation and consider all the initial conditions,

$$0 = (ms^2 + k_y)Y - msy(0) - m\dot{y}(0) \quad (12)$$

$$Y = \frac{msy(0)+m\dot{y}(0)}{ms^2+k_y} = \frac{-0.8s}{8s^2+6400} = \frac{-0.1s}{s^2+800} \quad (13)$$

$$y(t) = -0.1cos(20\sqrt{2}t) \quad (14)$$

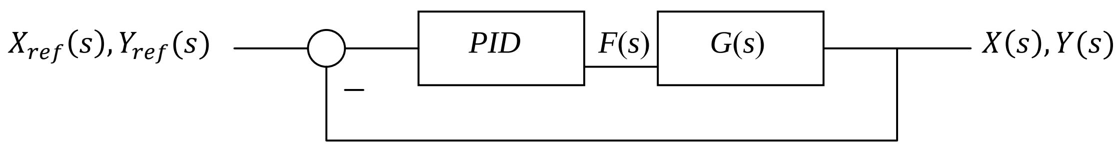
Thus the mass coordinate as a function of time,

$$(x(t), y(t)) = (0.05cos(10\sqrt{5}t), -0.1cos(20\sqrt{2}t)) \quad \underline{(15)}$$

$$\omega_x = 10\sqrt{5} = 22.36rad/sec = 3.56Hz \quad \underline{(16)}$$

$$\omega_y = 20\sqrt{2} = 28.28rad/sec = 4.50Hz \quad \underline{(17)}$$

(c) In order to make trajectory of the mass coordinate becomes elliptical shape without dimension change in x axis but dimension reduction in y axis, oscillation frequencies of both axes must be the same with no damping in x axis and with damping in y axis. PID controllers are applied to determine forces along x and y axes as shown in the below block diagram. Design all the gains of both axes that make the oscillation frequency in both axes become 1 Hz and time constant in y axis become 1 sec. Then determine steady-state response to a unit-step command in y axis. The gains can be positive, negative, or zero. (20)



Solution

Closed loop transfer function in x axis, $T_{cx}(s)$, is determined.

$$T_{cx}(s) = \frac{K_{Dx}s^2+K_{Px}s+K_{Ix}}{8s^3+K_{Dx}s^2+(K_{Px}+4000)s+K_{Ix}} \quad (18)$$

The desired characteristic equation in x axis,

$$s^2+(2\pi)^2 = s^2+39.48 = 0 \quad (19)$$

Thus,

$$K_{Ix} = 0 \quad (20)$$

$$K_{Dx} = 0 \quad (21)$$

$$K_{Px} = -3684.2 \quad (22)$$

Closed loop transfer function in y axis, $T_{cy}(s)$, is determined.

$$T_{cy}(s) = \frac{K_{Dy}s^2+K_{Py}s+K_{Iy}}{8s^3+K_{Dy}s^2+(K_{Py}+6400)s+K_{Iy}} \quad (23)$$

The desired characteristic equation in y axis,

$$(s+1+2\pi j)(s+1-2\pi j) = s^2+2s+((2\pi)^2+1) = s^2+2s+40.48 = 0 \quad (24)$$

Thus,

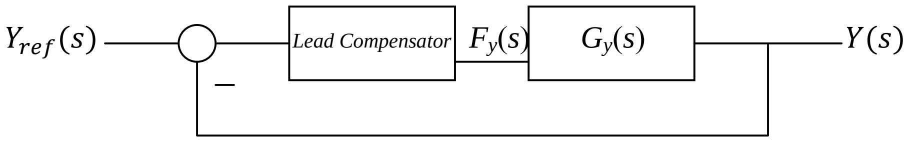
$$K_{Iy} = 0 \quad (25)$$

$$K_{Dy} = 16 \quad (26)$$

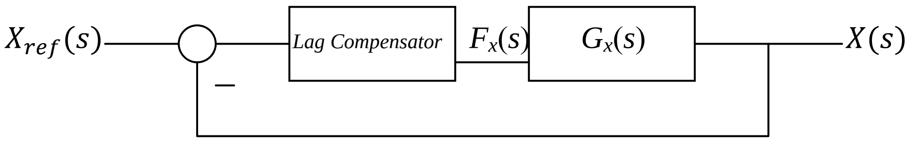
$$K_{Py} = -6076.2 \quad (27)$$

$$y_{ss} = \frac{K_{Py}}{(K_{Py}+6400)} = -18.77 \quad (28)$$

(d) If a lead compensator replaces PID controller along y axis to control the mass coordinate as shown in the block diagram (a), design all the parameters of the lead compensator and its gain that make the oscillation frequency become 1 Hz and time constant in y axis become 1 sec with the remaining pole at -10. What is the steady-state response to a unit-step input in y axis? If a lag compensator with $T = 1000, K = 3000, a = 0.1$ replaces PID controller along x axis as shown in the block diagram (b), determine the characteristic equation and the steady-state response to a unit-step input in x axis? (20)



(a)



(b)

Solution

Closed loop transfer function in y axis, $T_{cy}(s)$, is determined.

$$T_{cy}(s) = \frac{Ks+\frac{K}{aT}}{8s^3+\left(\frac{8}{T}\right)s^2+(K+6400)s+\frac{K}{aT}+\frac{6400}{T}} \quad (29)$$

Characteristic equation,

$$8s^3 + \left(\frac{8}{T}\right)s^2 + (K+6400)s + \frac{K}{aT}+\frac{6400}{T} = s^3 + \left(\frac{1}{T}\right)s^2 + \left(\frac{K}{8}+800\right)s + \frac{K}{8aT}+\frac{800}{T} = 0 \quad (30)$$

The desired characteristic equation,

$$(s^2+2s+40.48)(s+10) = s^3+12s^2+60.48s+404.8 = 0 \quad (31)$$

$$T = \frac{1}{12} = 0.083 \quad (32)$$

$$K = -5916.2 \quad (33)$$

$$a = 0.97 \quad (34)$$

$$y_{ss} = \frac{\frac{K}{aT}}{\frac{K}{aT}+\frac{6400}{T}} = -20.27 \quad (35)$$

Closed loop transfer function in x axis, $T_{cx}(s)$, is determined.

$$T_{cx}(s) = \frac{Kas+\frac{K}{T}}{8s^3+\left(\frac{8}{T}\right)s^2+(Ka+4000)s+\frac{K}{T}+\frac{4000}{T}} \quad (36)$$

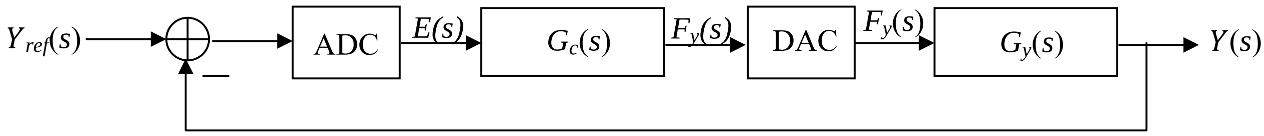
Characteristic equation,

$$8s^3 + \left(\frac{8}{T}\right)s^2 + (Ka+4000)s + \frac{K}{T}+\frac{4000}{T} = s^3 + \left(\frac{1}{T}\right)s^2 + \left(\frac{Ka}{8}+500\right)s + \frac{K}{8T}+\frac{500}{T} = 0 \quad (37)$$

$$s^3 + 0.001s^2 + 537.5s + 0.875 = 0 \quad (38)$$

$$x_{ss} = \frac{\frac{K}{T}}{\frac{K}{T}+\frac{4000}{T}} = 0.43 \quad (39)$$

(e) If a digital controller, $G_c(s)$, as shown in the block diagram below is used to control the mass coordinate in y axis, design the controller by direct design method when the desired closed loop transfer function, T_c, is represented in s domain by $\frac{1-e^{-Ts}}{s}\cdot\frac{1}{s+1}$. The sampling time, T, is 0.2 second. Then determine the control signal at step k, $f(k)$, as a function of control signal and error, e, at the current and previous steps. (20)



<u>Solution</u>

Firstly determine plant in combination with zero-order hold circuit DAC,

$$T_{c2}(s)=\frac{T_c}{s}=\frac{1}{s}\cdot\frac{1}{s+1}=\frac{1}{s}-\frac{1}{s+1} \tag{40}$$

$$T_{c2}(z)=\frac{z}{z-1}-\frac{z}{z-e^{-0.2}}=\frac{z}{z-1}-\frac{z}{z-0.82} \tag{41}$$

$$T_c(z)=\frac{z-1}{z}G_2(z)=1-\frac{z-1}{z-0.82}=\frac{0.18}{z-0.82} \tag{42}$$

$$G_{y2}(s)=\frac{1}{s}\cdot\frac{1}{8(s^2+800)}=\frac{1}{6400s}-\frac{s}{6400(s^2+800)} \tag{43}$$

$$G_{y2}(z)=\frac{1}{6400}\left(\frac{z}{z-1}-\frac{z\left(z-cos\left(4\sqrt{2}\right)\right)}{\left(z^2-2zcos\left(4\sqrt{2}\right)+1\right)}\right)=\frac{1}{6400}\left(\frac{z}{z-1}-\frac{z^2-0.81z}{z^2-1.62z+1}\right) \tag{44}$$

$$G_y(z)=\frac{z-1}{z}G_{y2}(z)=\frac{1}{6400}\left(1-\frac{z^2-1.81z+0.81}{z^2-1.62z+1}\right)=\frac{1}{6400}\left(\frac{0.19z+0.19}{z^2-1.62z+1}\right) \tag{45}$$

$$G_c=\frac{T_c}{G_y(1-T_c)}=\frac{6400\frac{0.18}{z-0.82}}{\left(\frac{0.19z+0.19}{z^2-1.62z+1}\right)\left(1-\frac{0.18}{z-0.82}\right)}=\frac{6400\frac{0.18}{z-0.82}}{\left(\frac{0.19z+0.19}{z^2-1.62z+1}\right)\left(\frac{z-1}{z-0.82}\right)}=\frac{6063.2\left(z^2-1.62z+1\right)}{\left(z^2-1\right)} \tag{46}$$

$$G_c=\frac{F}{E}=\frac{6063.2z^2-9822.3z+6063.2}{z^2-1} \tag{47}$$

$$(z^2-1)F=(6063.2z^2-9822.3z+6063.2)E \tag{48}$$

$$\left(1-\frac{1}{z^2}\right)F=\left(6063.2-\frac{9822.3}{z}+\frac{6063.2}{z^2}\right)E \tag{49}$$

$$f(k)=f(k-2)+6063.2e(k)-9822.3e(k-1)+6063.2e(k-2) \tag{50}$$