Final Examination **Control Theory AT74.02** **November 30, 2016**

Time: 13:00-15:00 hrs. Open Book

Marks: 100

Attempt all questions.

Consider again the cruise control system of a passenger car when throttle valve angle, θ, is used to control speed of the car, v. The relation between throttle valve angle (in degree) and car speed (in km/h) is expressed by the following transfer function

$$\frac{v}{\theta} = \frac{2}{s^2 + 5.1s + 0.5}$$

(a) Determine a state-space representation of the system when the state variable x_1 represents the car distance (d), the state variable x_2 represents the car speed, x_3 represents the car acceleration, the control signal is the throttle valve angle, θ. Both car distance and car speed are directly measured from the sensors. (25)

Solution

$$\frac{v}{\theta} = \frac{sd}{\theta} = \frac{2}{s^2+5.1s+0.5} \cdot \frac{(1000\, meters)}{(3600\, seconds)} = \frac{0.56}{s^2+5.1s+0.5} \quad (1)$$

$$\frac{d}{\theta} = \frac{0.56}{s^3+5.1s^2+0.5s} \quad (2)$$

$$x_1 = d \quad (3)$$

$$x_2 = \dot{d} \quad (4)$$

$$x_3 = \ddot{d} \quad (5)$$

Thus,

$$\dot{x}_1 = x_2 \quad (6)$$

$$\dot{x}_2 = x_3 \quad (7)$$

From (2),

$$\dddot{d} + 5.1\ddot{d} + 0.5\dot{d} = 0.56\theta \quad (8)$$

$$\dot{x}_3 = -0.5x_2 - 5.1x_3 + 0.56\theta \quad (9)$$

Thus,

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.5 & -5.1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ 0.56\end{bmatrix}[\theta] \quad \underline{(10)}$$

$$\begin{bmatrix}y_1 \\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} \quad \underline{(11)}$$

(b) If the car is controlled to stop at a distance, d_r, remodel the state-space system by taking into consideration the references. (25)

Solution

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d = Ae + Bu + Ex_0 \quad (1)$$

When

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} d - d_r \\ \dot{d} - 0 \\ \ddot{d} - 0 \end{bmatrix} \quad (2)$$

$$A - A_r = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.5 & -5.1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.5 & -5.1 \end{bmatrix} \quad (3)$$

Thus,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.5 & -5.1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.56 \end{bmatrix} [\theta] + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.5 & -5.1 \end{bmatrix} \begin{bmatrix} d_r \\ 0 \\ 0 \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.5 & -5.1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.56 \end{bmatrix} [\theta] \quad \underline{(5)}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \quad \underline{(6)}$$

(c) If the cost function is expressed by $V = \int_0^\infty (10(d - d_r)^2 + \theta^2)dt$, determine the throttle valve angle, θ, as the function of the state errors that minimizes the cost function. What are the optimal characteristic equation and its poles? (25)

Solution

$$Q = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1)$$

$$R = [1] \quad (2)$$

Control signal is determined from

$$\theta = -R^{-1}B^t\bar{M}e \quad (3)$$

$$R^{-1}B^t\bar{M} = [1]^{-1} \begin{bmatrix} 0 \\ 0 \\ 0.56 \end{bmatrix}^t \begin{bmatrix} m_1 & m_2 & m_3 \\ m_2 & m_4 & m_5 \\ m_3 & m_5 & m_6 \end{bmatrix} \quad (4)$$

$$R^{-1}B^t\bar{M} = [0.56m_3 \quad 0.56m_5 \quad 0.56m_6] \quad (5)$$

When

$$0 = -\dot{\bar{M}} = \bar{M}A + A^t\bar{M} - \bar{M}BR^{-1}B^t\bar{M} + Q \quad (6)$$

Substitute all the concerned matrices,

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}m_1&m_2&m_3\\m_2&m_4&m_5\\m_3&m_5&m_6\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\0&-0.5&-5.1\end{bmatrix}+\begin{bmatrix}0&0&0\\1&0&-0.5\\0&1&-5.1\end{bmatrix}\begin{bmatrix}m_1&m_2&m_3\\m_2&m_4&m_5\\m_3&m_5&m_6\end{bmatrix}$$
$$-\begin{bmatrix}m_1&m_2&m_3\\m_2&m_4&m_5\\m_3&m_5&m_6\end{bmatrix}\begin{bmatrix}0\\0\\0.56\end{bmatrix}\begin{bmatrix}0.56m_3&0.56m_5&0.56m_6\end{bmatrix}+\begin{bmatrix}10&0&0\\0&0&0\\0&0&0\end{bmatrix} \quad (7)$$

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}-0.31m_3^2+10 & m_1-0.5m_3-0.31m_3m_5 & m_2-5.1m_3-0.31m_3m_6\\ m_1-0.5m_3-0.31m_3m_5 & 2m_2-m_5-0.31m_5^2 & m_3+m_4-5.1m_5-0.5m_6-0.31m_5m_6\\ m_2-5.1m_3-0.31m_3m_6 & m_3+m_4-5.1m_5-0.5m_6-0.31m_5m_6 & 2m_5-10.2m_6-0.31m_6^2\end{bmatrix} \quad (8)$$

Thus,

$$\begin{bmatrix}m_1&m_2&m_3\\m_2&m_4&m_5\\m_3&m_5&m_6\end{bmatrix}=\begin{bmatrix}25.85&33.01&5.65\\33.01&71.54&13.00\\5.65&13.00&2.38\end{bmatrix} \quad (9)$$

$$R^{-1}B^t\bar{M}=\begin{bmatrix}3.16&7.28&1.33\end{bmatrix} \quad (10)$$

$$\theta=-3.16e_1-7.28e_2-1.33e_3 \quad \underline{(11)}$$

$$|sI-A_c|=\left|\begin{bmatrix}s&0&0\\0&s&0\\0&0&s\end{bmatrix}-\begin{bmatrix}0&1&0\\0&0&1\\0&-0.5&-5.1\end{bmatrix}+\begin{bmatrix}0\\0\\0.56\end{bmatrix}\begin{bmatrix}3.16&7.28&1.33\end{bmatrix}\right|=\begin{vmatrix}s&-1&0\\0&s&-1\\1.77&4.58&s+5.84\end{vmatrix} \quad (12)$$

$$|sI-A_c|=s^3+5.84s^2+4.58s+1.77=(s+5.00)(s+0.42+0.42j)(s+0.42-0.42j)=0 \quad \underline{(13)}$$

(d) The reduced-order observer is designed to place the pole at -10. Write the equations that are used to determine or estimate all the state errors. (25)

<u>Solution</u>

$$\begin{bmatrix}\dot{e}_1\\\dot{e}_2\\\dot{e}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\0&-0.5&-5.1\end{bmatrix}\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}+\begin{bmatrix}0\\0\\0.56\end{bmatrix}[\theta] \quad (1)$$

$$\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix} \quad (2)$$

Design the reduced-order observer,

$$x_1=\begin{bmatrix}e_1\\e_2\end{bmatrix}, x_2=[e_3], C_1=\begin{bmatrix}1&0\\0&1\end{bmatrix}, C_2=\begin{bmatrix}0\\0\end{bmatrix},$$

$$A_{11}=\begin{bmatrix}0&1\\0&0\end{bmatrix}, A_{12}=\begin{bmatrix}0\\1\end{bmatrix}, A_{21}=\begin{bmatrix}0&-0.5\end{bmatrix}, A_{22}=[-5.1], B_1=\begin{bmatrix}0\\0\end{bmatrix}, B_2=[0.56] \quad (3)$$

$$\hat{x}_1=\begin{bmatrix}e_1\\e_2\end{bmatrix} \quad \underline{(4)}$$

$$\hat{x}_2=Ly+z \quad (5)$$

$$\dot{z}=F\hat{x}_2+\bar{g}y+Hu \quad (6)$$

From the reduced order observer gains requirement,

$$F=A_{22}-LC_1A_{12}=[-5.1]-\begin{bmatrix}l_1&l_2\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=[-5.1-l_2] \quad (7)$$

$$|sI-F|=|s+5.1+l_2|=s+10 \quad (8)$$

$$[l_1 \quad l_2] = [0 \quad 4.9] \tag{9}$$

$$F = [-10] \tag{10}$$

$$\bar{g} = (A_{21} - LC_1A_{11})C_1^{-1} = \left([0 \quad -0.5] - [0 \quad 4.9]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^{-1} = [0 \quad -0.5] \tag{11}$$

$$H = B_2 - LC_1B_1 = [0.56] - [0 \quad 4.9]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = [0.56] \tag{12}$$

Substitute all the concerned matrices into (5) and (6),

$$[\hat{e}_3] = [0 \quad 4.9]\begin{bmatrix}y_1\\y_2\end{bmatrix} + z \tag{13}$$

$$\dot{z} = [-10][\hat{e}_3] + [0 \quad -0.5]\begin{bmatrix}y_1\\y_2\end{bmatrix} + [0.56][\theta] \tag{14}$$