Midterm Examination **Control Theory AT74.02** **September 27, 2016**

Time: 10:00-12:00 hrs. Open Book

Marks: 100

Attempt all questions.

In cruise control system of a passenger car, throttle valve angle, θ, is used to control speed of the car, v. The relation between throttle valve angle (in degree) and car speed (in km/h) is expressed by the following transfer function

$$\frac{v}{\theta} = \frac{2}{s^2 + 5.1s + 0.5}$$

(a) Determine the speed of the car, v, from the rest position (in km/h) as a function of time, t, (in seconds), if the throttle valve angle is opened as expressed by the equation (25)

$$\theta(t) = \begin{cases} 20t & ; 0 < t < 2 \\ 40 & ; 2 \le t \end{cases}$$

Solution

From the transfer function, the original differential equation is

$$\ddot{v} + 5.1\dot{v} + 0.5v = 2\theta$$

By taking Laplace Transformation,

$$[s^2 + 5.1s + 0.5]v = sv(0) + \dot{v}(0) + 5.1v(0) + 2\theta$$

$$v(s) = \frac{sv(0) + \dot{v}(0) + 5.1v(0) + 2\theta}{s^2 + 5.1s + 0.5}$$

During the first two seconds,

$$\theta(s) = \frac{20}{s^2}$$

$$v(s) = \frac{2}{s^2 + 5.1s + 0.5} \cdot \frac{20}{s^2} = \frac{A}{s + 0.1} + \frac{B}{s + 5} + \frac{Cs + D}{s^2}$$

$$As^3 + 5As^2 + Bs^3 + 0.1Bs^2 + Cs^3 + 5.1Cs^2 + 0.5Cs + Ds^2 + 5.1Ds + 0.5D = 40$$

$$A = 815.67$$

$$B = -0.33$$

$$C = -816$$

$$D = 80$$

$$v(s) = \frac{816.33}{s + 0.1} - \frac{0.33}{s + 5} + \frac{-816s + 80}{s^2}$$

$$v(t) = 816.33e^{-0.1t} - 0.33e^{-5t} - 816 + 80t$$

$$v(2) = 816.33e^{-0.2} - 0.33e^{-10} - 816 + 160 = 12.35$$

$$\dot{v}(t) = -81.63e^{-0.1t} + 1.56e^{-5t} + 80$$

$$\dot{v}(2) = -81.63e^{-0.2} + 1.56e^{-10} + 80 = 13.17$$

After two seconds,

$$\theta(s) = \frac{40}{s}$$

$$v(s) = \frac{sv(2) + \dot{v}(2) + 5.1v(2)}{s^2 + 5.1s + 0.5} + \frac{2\theta}{s^2 + 5.1s + 0.5}$$

$$v(s) = \frac{12.35s + 13.17 + 5.1(12.35)}{s^2 + 5.1s + 0.5} + \frac{2}{s^2 + 5.1s + 0.5} \cdot \frac{40}{s}$$

$$v(s) = \frac{12.35s^2 + 76.16s + 80}{(s^2 + 5.1s + 0.5)s} = \frac{A}{s + 0.1} + \frac{B}{s + 5} + \frac{C}{s}$$

$$As^2 + 5As + Bs^2 + 0.1Bs + Cs^2 + 5.1Cs + 0.5C = 12.35s^2 + 76.16s + 80$$

$$A = -147.97$$

$$B = 0.32$$

$$C = 160$$

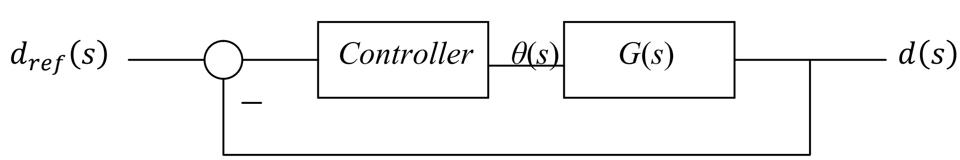
$$v(s) = \frac{-147.97}{s + 0.1} + \frac{0.32}{s + 5} + \frac{160}{s}$$

$$v(t) = -147.97e^{-0.1(t-2)} + 0.32e^{-5(t-2)} + 160$$

Thus,

$$v(t) = \begin{cases} 816.33e^{-0.1t} - 0.33e^{-5t} - 816 + 80t & ; 0 < t < 2 \\ -147.97e^{-0.1(t-2)} + 0.32e^{-5(t-2)} + 160 & ; 2 \le t \end{cases}$$

(b) In order to control the car position, PD controller is applied as shown in the below block diagram. Determine the new transfer function, G, from throttle valve angle (in degree), θ, to car distance (in meter), d. Determine the gains that make the dominant time constant of the controlled system become 5 seconds and damping ratio of 0.8. Determine the location of the remaining pole of the controlled system and also the steady-state distance when the reference distance is 200 meters. (25)



Solution

The transfer function from throttle valve angle to car distance (in meter) is determined.

$$\frac{v}{\theta} = \frac{sd}{\theta} = \frac{2}{s^2 + 5.1s + 0.5} \cdot \frac{(1000\ meters)}{(3600\ seconds)} = \frac{0.56}{s^2 + 5.1s + 0.5}$$

$$G = \frac{d}{\theta} = \frac{0.56}{s^3 + 5.1s^2 + 0.5s}$$

The closed loop transfer function with PD controller is then determined.

$$G_c = \frac{d}{d_{ref}} = \frac{0.56K_D s + 0.56K_P}{s^3 + 5.1s^2 + (0.56K_D + 0.5)s + 0.56K_P}$$

Characteristic of the dominant root makes 5 seconds time constant and 0.8 damping ratio.

$$s^2 + 0.4s + 0.0625 = 0$$

The desired characteristic equations follows

$$(s^2 + 0.4s + 0.0625)(s + x) = s^3 + (0.4 + x)s^2 + (0.0625 + 0.4x)s + 0.0625x = 0$$

$$s^3 + (0.4 + x)s^2 + (0.0625 + 0.4x)s + 0.0625x = s^3 + 5.1s^2 + (0.56K_D + 0.5)s + 0.56K_P$$

$$x = 4.7$$

$$K_P = 0.52$$

$$K_D = 2.58$$

The steady-state distance with 200 meters reference distance is determined from final value theorem.

$$d_{ss} = \lim_{s \to 0} s \frac{0.56K_D s + 0.56K_P}{s^3 + 5.1s^2 + (0.56K_D + 0.5)s + 0.56K_P} \frac{200}{s} = 200$$

(c) If a lead compensator, expressed by $K\frac{(s+z)}{(s+p)}$ with $K = 1$, replaces the PD controller in (b) in order to control the car position, determine the zero, *z*, and the pole, *p*, of the lead compensator, that make the dominant time constant of the controlled system become 5 seconds and damping ratio of 0.8. Determine the locations of the remaining two poles of the controlled system and also the steady-state distance when the reference distance is 200 meters. (25)

Solution

The closed loop transfer function with lead compensator is then determined.

$$G_c = \frac{d}{d_{ref}} = \frac{0.56Ks + 0.56Kz}{s^4 + (5.1 + p)s^3 + (0.5 + 5.1p)s^2 + (0.56K + 0.5p)s + 0.56Kz}$$

With $K = 1$,

$$G_c = \frac{d}{d_{ref}} = \frac{0.56s + 0.56z}{s^4 + (5.1 + p)s^3 + (0.5 + 5.1p)s^2 + (0.56 + 0.5p)s + 0.56z}$$

Characteristic of the dominant root makes 5 seconds time constant and 0.8 damping ratio.

$$s^2 + 0.4s + 0.0625 = 0$$

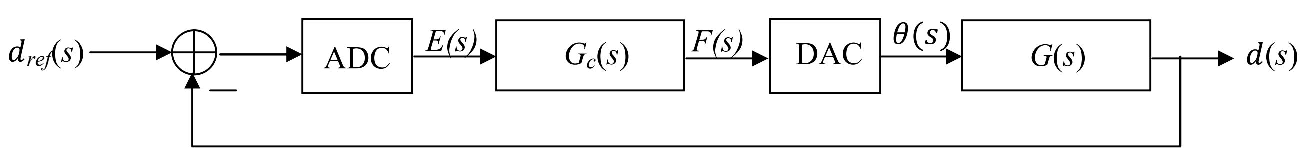
The desired characteristic equations follows

$$(s^2+0.4s+0.0625)(s^2+(x+y)s+xy)$$
$$= s^4+\big(0.4+(x+y)\big)s^3+(0.0625+0.4(x+y)+xy)s^2+(0.0625(x+y)+0.4xy)s$$
$$+0.0625xy = 0$$
$$s^4+(5.1+p)s^3+(0.5+5.1p)s^2+(0.56+0.5p)s+0.56z$$
$$= s^4+\big(0.4+(x+y)\big)s^3+(0.0625+0.4(x+y)+xy)s^2+(0.0625(x+y)+0.4xy)s$$
$$+0.0625xy$$
$$z = 0.15$$
$$p = 0.58$$
$$x = 5.02$$
$$y = 0.26$$

The steady-state distance with 200 meters reference distance is determined from final value theorem.

$$d_{ss} = \lim_{s\to 0} s\frac{0.56s+0.56z}{s^4+(5.1+p)s^3+(0.5+5.1p)s^2+(0.56+0.5p)s+0.56z}\frac{200}{s} = 200$$

(d) If a digital controller, $G_c(s)$, as shown in the block diagram below is used to control the car position, design the controller by direct design method when the desired closed loop transfer function, T_c, is represented in s domain by $\frac{1-e^{-Ts}}{s}\cdot\frac{0.0625}{s^2+0.4s+0.0625}$. The sampling time, T, is 0.1 second. Then determine the control signal at step k, $f(k)$, as a function of control signal and error, e, at the current and previous steps. (25)



Solution

Determine the required closed loop transfer function with zero-order hold circuit,

$$T_{c2}(s) = \frac{T_c}{s} = \frac{1}{s}\cdot\frac{0.0625}{s^2+0.4s+0.0625} = \frac{1}{s}-\frac{s+0.4}{s^2+0.4s+0.0625} = \frac{1}{s}-\frac{(s+0.2)+1.33\times 0.15}{(s+0.2)^2+0.15^2}$$

$$T_{c2}(z) = \frac{z}{z-1}-\frac{z^2-2e^{-0.02}cos(0.015)+1.33\times ze^{-0.02}sin(0.015)}{z^2-2ze^{-0.02}cos(0.015)+e^{-0.04}}$$

$$T_{c2}(z) = \frac{z}{z-1}-\frac{z^2+0.02z-1.96}{z^2-1.96z+0.96}$$

$$T_c(z) = \frac{z-1}{z}T_{c2}(z) = 1-\frac{z^3-0.98z^2-1.98z+1.96}{z^3-1.96z^2+0.96z} = \frac{-0.98z^2+2.94z-1.96}{z^3-1.96z^2+0.96z}$$

Determine the plant transfer function with zero-order hold circuit,

$$G_2(s)=\frac{1}{s}\cdot\frac{0.56}{s^3+5.1s^2+0.5s}=\frac{-11.42s+1.12}{s^2}-\frac{0.00}{s+5}+\frac{11.43}{s+0.1}$$

$$G_2(z)=\frac{-11.42z}{z-1}+\frac{0.11z}{(z-1)^2}+\frac{11.43z}{z-e^{-0.01}}$$

$$G(z)=\frac{z-1}{z}G_2(z)=-11.42+\frac{0.11}{z-1}+\frac{11.43z-11.43}{z-0.99}$$

$$G_c=\frac{T_c}{G(1-T_c)}=\frac{\frac{-0.98z^2+2.94z-1.96}{z^3-1.96z^2+0.96z}}{\left(-11.42+\frac{0.11}{z-1}+\frac{11.43z-11.43}{z-0.99}\right)\left(1-\frac{-0.98z^2+2.94z-1.96}{z^3-1.96z^2+0.96z}\right)}$$

$$G_c=\frac{F}{E}=\frac{-0.98z^4+4.89z^3-8.78z^2+6.81z-1.94}{0.01z^5-0.03z^4+0.01z^3+0.05z^2-0.06z+0.02}$$

$$\begin{aligned}&(0.01z^5-0.03z^4+0.01z^3+0.05z^2-0.06z+0.02)F\\&\qquad=(-0.98z^4+4.89z^3-8.78z^2+6.81z-1.94)E\end{aligned}$$

$$\left(1-\frac{3}{z}+\frac{1}{z^2}+\frac{5}{z^3}-\frac{6}{z^4}+\frac{2}{z^5}\right)F=\left(-\frac{98}{z}+\frac{489}{z^2}-\frac{878}{z^3}+\frac{681}{z^4}-\frac{194}{z^5}\right)E$$

$$\begin{aligned}f(k)=&\,3f(k-1)-f(k-2)-5f(k-3)+6f(k-4)-2f(k-5)-98e(k-1)+489e(k-2)\\&-878e(k-3)+681e(k-4)-194e(k-5)\end{aligned}$$