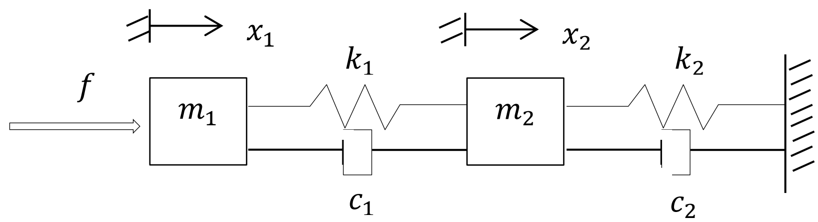
Final Examination **Control Theory AT74.10** **November 22, 2018**

Time: 9:00-11:00 hrs. Open Book
Marks: 100

Attempt all questions.

Consider the system given in the midterm examination again. When an external force, f, is applied to a 2-mass-2-spring-2-damper system as shown in the below figure. x_1 and x_2 are the distances of m_1 and m_2 from the fixed points. Both distances are 0 at the steady state when there is no external force applied. Use $m_1 = 0.5$ kg, $m_2 = 2$ kg, $k_1 = 2000$ N/m, $k_2 = 500$ N/m, $c_1 = 20$ Ns/m, $c_2 = 4$ Ns/m.



The system is considered in the first-companion form in questions (a)-(c) and considered in the second-companion form in questions (d)-(f).

(a) Determine a state-space representation of the system in the first-companion form when the state variables are numbered from right to left with the input, u, of the external force, f, and the output, y, of the distance, x_2. (10)

Solution

Transfer function from the input of the external force, f, to the output of the distance, x_2,

$$G = \frac{X_2}{F} = \frac{c_1 s + k_1}{m_1 m_2 s^4 + (m_1(c_1+c_2) + m_2 c_1)s^3 + (m_1(k_1+k_2) + m_2 k_1 + c_1 c_2)s^2 + (c_1 k_2 + c_2 k_1)s + k_1 k_2} \quad (1)$$

Substitute all the parameters,

$$\frac{X_2}{F} = \frac{20s+2000}{s^4+52s^3+5330s^2+18000s+1000000} \quad (2)$$

Thus,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1000000 & -18000 & -5330 & -52\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}[u] \quad \underline{(3)}$$

$$[y] = [2000 \quad 20 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \quad \underline{(4)}$$

(b) If the output is controlled to a constant reference distance, y_r, determine the reference state vector. Then remodel the state-space system in (a) by taking into consideration the reference. (10)

Solution

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d = Ae + Bu + Ex_0 \tag{1}$$

For constant reference distance of the first-companion from,

$$\begin{bmatrix} x_{1r} \\ x_{2r} \\ x_{3r} \\ x_{4r} \end{bmatrix} = \begin{bmatrix} x_{1r} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

$$[y_r] = \begin{bmatrix} 2000 & 20 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1r} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} x_{1r} \\ x_{2r} \\ x_{3r} \\ x_{4r} \end{bmatrix} = \begin{bmatrix} \frac{y_r}{2000} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} x_1 - \frac{y_r}{2000} \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{5}$$

$$E = A - A_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1000000 & -18000 & -5330 & -52 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1000000 & -18000 & -5330 & -52 \end{bmatrix} \tag{6}$$

Thus,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1000000 & -18000 & -5330 & -52 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

$$+ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1000000 & -18000 & -5330 & -52 \end{bmatrix} \begin{bmatrix} x_{1r} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

$$[y] = \begin{bmatrix} 2000 & 20 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \tag{8}$$

(c) Design the regulator for the system in (b) by placing the dominant poles to have the time constant of 0.25 s with frequency of 25 rad/s and the remaining poles at $-25 \pm 25j$. No output steady-state error from the constant reference distance is required. (20)

Solution

Design the regulator,

$|sI - A_c| = |sI - A + BG|$

$$= \left| \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1000000 & -18000 & -5330 & -52 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} [g_{11} \quad g_{12} \quad g_{13} \quad g_{14}] \right|$$

$$= \left\| \begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & -1 \\ 1000000 + g_{11} & 18000 + g_{12} & 5330 + g_{13} & s + 52 + g_{14} \end{bmatrix} \right\| = 0 \quad (1)$$

$$|sI - A_c| = s^4 + (52 + g_{14})s^3 + (5330 + g_{13})s^2 + (18000 + g_{12})s + 1000000 + g_{11} = 0 \quad (2)$$

The desired characteristic equations

$$(s + 4 + 25j)(s + 4 - 25j)(s + 25 + 25j\,)(s + 25 - 25j\,) = (s^2 + 8s + 641)(s^2 + 50s + 1250)$$

$$= s^4 + 58s^3 + 2291s^2 + 42050s + 801250 = 0 \quad (3)$$

$$[g_{11} \quad g_{12} \quad g_{13} \quad g_{14}] = [-198750 \quad 24050 \quad -3039 \quad 6] \quad \underline{(4)}$$

The gain for the reference,

$$g_0 = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1}E \quad (5)$$

$$A - BG = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1000000 - g_{11} & -18000 - g_{12} & -5330 - g_{13} & -52 - g_{14} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -801250 & -42050 & -2291 & -58 \end{bmatrix} \quad (6)$$

$$[A - BG]^{-1} = \begin{bmatrix} -0.0525 & -0.0029 & -0.0001 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (7)$$

$$C[A - BG]^{-1} = [2000 \quad 20 \quad 0 \quad 0] \begin{bmatrix} -0.0525 & -0.0029 & -0.0001 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= [-84.9610 \quad -5.7186 \quad -0.1448 \quad -0.0025] \quad (8)$$

$$C[A-BG]^{-1}B = [-84.9610 \quad -5.7186 \quad -0.1448 \quad -0.0025]\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = [-0.0025] \qquad (9)$$

$$[C[A-BG]^{-1}B]^{-1} = [-400] \qquad (10)$$

$$g_0 = [C[A-BG]^{-1}B]^{-1}C[A-BG]^{-1}E$$

$$= [-400][-84.9610 \quad -5.7186 \quad -0.1448 \quad -0.0025]\begin{bmatrix}0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\-1000000 & -18000 & -5330 & -52\end{bmatrix}$$

$$= [-1000000 \quad 16012 \quad -3034 \quad -6] \qquad \underline{(11)}$$

Thus,

$$f = u = -[-198750 \quad 24050 \quad -3039 \quad 6]\begin{bmatrix}x_1 - \frac{y_r}{2000}\\x_2\\x_3\\x_4\end{bmatrix} - [-1000000 \quad 16012 \quad -3034 \quad -6]\begin{bmatrix}\frac{y_r}{2000}\\0\\0\\0\end{bmatrix}$$

$$= 198750\left(x_1 - \frac{y_r}{2000}\right) - 24050x_2 + 3039x_3 - 6x_4 + 500y_r \qquad (\underline{12})$$

(d) Determine a state-space representation of the system in the second-companion form when the state variables are numbered from right to left with the input, u, of the external force, f, and the output, y, of the distance, x_2. (10)

<u>Solution</u>

Transfer function from the input of the external force, f, to the output of the distance, x_2,

$$G = \frac{X_2}{F} = \frac{c_1 s + k_1}{m_1 m_2 s^4 + (m_1(c_1+c_2)+m_2 c_1)s^3 + (m_1(k_1+k_2)+m_2 k_1 + c_1 c_2)s^2 + (c_1 k_2 + c_2 k_1)s + k_1 k_2} \qquad (1)$$

Substitute all the parameters,

$$\frac{X_2}{F} = \frac{20s+2000}{s^4+52s^3+5330s^2+18000s+1000000} \qquad (2)$$

Thus,

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\end{bmatrix} = \begin{bmatrix}-52 & 1 & 0 & 0\\-5330 & 0 & 1 & 0\\-18000 & 0 & 0 & 1\\-1000000 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \begin{bmatrix}0\\0\\20\\2000\end{bmatrix}[u] \qquad \underline{(3)}$$

$$[y] = [1 \quad 0 \quad 0 \quad 0]\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \qquad \underline{(4)}$$

(e) If the output is controlled to a constant reference distance, y_r, determine the reference state vector. Then remodel the state-space system in (d) by taking into consideration the reference. (20)

<u>Solution</u>

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d = Ae + Bu + Ex_0 \tag{1}$$

For constant reference distance of the second-companion from,

$$[y_r] = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \\ x_{3r} \\ x_{4r} \end{bmatrix} = [x_{1r}] \tag{2}$$

From the dynamics of the states,

$$[\dot{x}_{1r}] = [0] = \begin{bmatrix} -52 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \\ x_{3r} \\ x_{4r} \end{bmatrix} + [0][u] \tag{3}$$

$$[x_{2r}] = [52x_{1r}] = [52y_r] \tag{4}$$

$$[\dot{x}_{2r}] = [0] = \begin{bmatrix} -5330 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \\ x_{3r} \\ x_{4r} \end{bmatrix} + [0][u] \tag{5}$$

$$[x_{3r}] = [5330x_{1r}] = [5330y_r] \tag{6}$$

$$[\dot{x}_{3r}] = [0] = \begin{bmatrix} -18000 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \\ x_{3r} \\ x_{4r} \end{bmatrix} + [20][u] \tag{7}$$

$$[x_{4r}] = [18000x_{1r}] - [20u] = [18000y_r - 20u] \tag{8}$$

At steady-state $\dot{u} = 0$,

$$[\dot{x}_{4r}] = [0] = \begin{bmatrix} -1000000 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \\ x_{3r} \\ x_{4r} \end{bmatrix} + [2000][u_{ss}] \tag{9}$$

$$[1000000x_{1r}] = [2000u_{ss}] \tag{10}$$

$$[u_{ss}] = [500x_{1r}] = [500y_r] \tag{11}$$

Substitute (11) into (8),

$$[x_{4r}] = [18000y_r - 20u] = [18000y_r - 10000y_r] = [8000y_r] \tag{12}$$

Thus,

$$\begin{bmatrix} x_{1r} \\ x_{2r} \\ x_{3r} \\ x_{4r} \end{bmatrix} = \begin{bmatrix} y_r \\ 52y_r \\ 5330y_r \\ 8000y_r \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} x_1 - y_r \\ x_2 - 52y_r \\ x_3 - 5330y_r \\ x_4 - 8000y_r \end{bmatrix} \tag{14}$$

$$E = A - A_r = \begin{bmatrix} -52 & 1 & 0 & 0 \\ -5330 & 0 & 1 & 0 \\ -18000 & 0 & 0 & 1 \\ -1000000 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -52 & 1 & 0 & 0 \\ -5330 & 0 & 1 & 0 \\ -18000 & 0 & 0 & 1 \\ -1000000 & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

Thus,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} = \begin{bmatrix} -52 & 1 & 0 & 0 \\ -5330 & 0 & 1 & 0 \\ -18000 & 0 & 0 & 1 \\ -1000000 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 20 \\ 2000 \end{bmatrix} [u]$$

$$+ \begin{bmatrix} -52 & 1 & 0 & 0 \\ -5330 & 0 & 1 & 0 \\ -18000 & 0 & 0 & 1 \\ -1000000 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \\ x_{3r} \\ x_{4r} \end{bmatrix} \quad \underline{(16)}$$

$$[y] = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \quad \underline{(17)}$$

(f) Design the regulator of the system in (e) by placing the dominant poles to have the time constant of 0.25 s with frequency of 25 rad/s and the remaining poles at $-25 \pm 25j$. No output steady-state error from the constant reference distance is required. (30)

<u>Solution</u>

Design the regulator,

$|sI - A_c| = |sI - A + BG|$

$$= \left| \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -52 & 1 & 0 & 0 \\ -5330 & 0 & 1 & 0 \\ -18000 & 0 & 0 & 1 \\ -1000000 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 20 \\ 2000 \end{bmatrix} [g_{11} \quad g_{12} \quad g_{13} \quad g_{14}] \right|$$

$$= \left| \begin{bmatrix} s+52 & -1 & 0 & 0 \\ 5330 & s & -1 & 0 \\ 18000 + 20g_{11} & 20g_{12} & s + 20g_{13} & -1 + 20g_{14} \\ 1000000 + 2000g_{11} & 2000g_{12} & 2000g_{13} & s + 2000g_{14} \end{bmatrix} \right| = 0 \quad (1)$$

$$|sI - A_c| = s^4 + (52 + 20g_{13} + 2000g_{14})s^3 + (5330 + 20g_{12} + 3040g_{13} + 104000g_{14})s^2$$
$$+ (18000 + 20g_{11} + 3040g_{12} + 210600g_{13} + 10660000g_{14})s$$
$$+1000000 + 2000g_{11} + 104000g_{12} + 10660000g_{13} + 16000000g_{14} = 0 \quad (2)$$

The desired characteristic equations

$$(s + 4 + 25j)(s + 4 - 25j)(s + 25 + 25j)(s + 25 - 25j) = (s^2 + 8s + 641)(s^2 + 50s + 1250)$$
$$= s^4 + 58s^3 + 2291s^2 + 42050s + 801250 = 0 \quad (3)$$

$$[g_{11} \quad g_{12} \quad g_{13} \quad g_{14}] = [8671.2 \quad -3.6 \quad -1.6 \quad 0] \quad \underline{(4)}$$

The gain for the reference,

$$g_0 = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1}E \quad (5)$$

$$A - BG = \begin{bmatrix} -52 & 1 & 0 & 0 \\ -5330 & 0 & 1 & 0 \\ -18000 - 20g_{11} & -20g_{12} & -20g_{13} & 1 - 20g_{14} \\ -1000000 - 2000g_{11} & -2000g_{12} & -2000g_{13} & -2000g_{14} \end{bmatrix}$$

$$= \begin{bmatrix} -52 & 1 & 0 & 0 \\ -5330 & 0 & 1 & 0 \\ -191424 & 71.02 & 32.80 & 0.61 \\ -18342000 & 7102.5 & 3280 & -38.80 \end{bmatrix} \quad (6)$$

$$[A - BG]^{-1} = \begin{bmatrix} 0.0089 & 0.0041 & -0.0000 & -0.0000 \\ 1.4609 & 0.2129 & -0.0025 & -0.0000 \\ 47.2465 & 22.8187 & -0.2581 & -0.0041 \\ 70.9141 & 32.7486 & 0.6126 & -0.0161 \end{bmatrix} \quad (7)$$

$$C[A - BG]^{-1} = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} 0.0089 & 0.0041 & -0.0000 & -0.0000 \\ 1.4609 & 0.2129 & -0.0025 & -0.0000 \\ 47.2465 & 22.8187 & -0.2581 & -0.0041 \\ 70.9141 & 32.7486 & 0.6126 & -0.0161 \end{bmatrix}$$

$$= [0.0089 \quad 0.0041 \quad -0.0000 \quad -0.0000] \quad (8)$$

$$C[A - BG]^{-1}B = [0.0089 \quad 0.0041 \quad -0.0000 \quad -0.0000] \begin{bmatrix} 0 \\ 0 \\ 20 \\ 2000 \end{bmatrix} = [-0.0025] \quad (9)$$

$$[C[A - BG]^{-1}B]^{-1} = [-400] \quad (10)$$

$$g_0 = [C[A - BG]^{-1}B]^{-1}C[A - BG]^{-1}E$$

$$= [-400][0.0089 \quad 0.0041 \quad -0.0000 \quad -0.0000] \begin{bmatrix} -52 & 1 & 0 & 0 \\ -5330 & 0 & 1 & 0 \\ -18000 & 0 & 0 & 1 \\ -1000000 & 0 & 0 & 0 \end{bmatrix}$$

$$= [8257.7 \quad -3.5 \quad -1.6 \quad 0] \quad \underline{(11)}$$

Thus,

$$f = u = -[8671.2 \quad -3.6 \quad -1.6 \quad 0] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} - [8257.7 \quad -3.5 \quad -1.6 \quad 0] \begin{bmatrix} y_r \\ 52y_r \\ 5330y_r \\ 8000y_r \end{bmatrix}$$

$$= -8671.2e_1 + 3.6e_2 + 1.6e_3 - 0e_4 + 500y_r \quad (\underline{12})$$