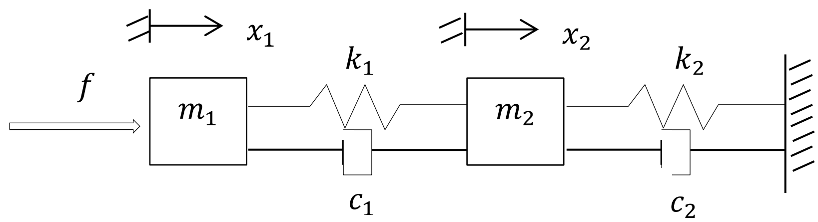
Midterm Examination **Control Theory AT74.10** **September 25, 2018**

Time: 9:00-11:00 hrs. Open Book

Marks: 100

Attempt all questions.

An external force, f, is applied to a 2-mass-2-spring-2-damper system as shown in the below figure. x_1 and x_2 are the distances of m_1 and m_2 from the fixed points. Both distances are 0 at the steady state when there is no external force applied. Use $m_1 = 0.5$ kg, $m_2 = 2$ kg, $k_1 = 2000$ N/m, $k_2 = 500$ N/m, $c_1 = 20$ Ns/m, $c_2 = 4$ Ns/m.



(a) Derive that the transfer function from the external force, f, to displacement of the second mass, x_2, is $G = \frac{X_2}{F} = \frac{c_1 s + k_1}{m_1 m_2 s^4 + (m_1(c_1+c_2)+m_2 c_1)s^3 + (m_1(k_1+k_2)+m_2 k_1 + c_1 c_2)s^2 + (c_1 k_2 + c_2 k_1)s + k_1 k_2}$. (10 Points)

Solution

$$f - k_1(x_1 - x_2) - c_1(\dot{x}_1 - \dot{x}_2) = m_1 \ddot{x}_1$$

$$k_1(x_1 - x_2) + c_1(\dot{x}_1 - \dot{x}_2) - k_2 x_2 - c_2 \dot{x}_2 = m_2 \ddot{x}_2$$

Take Laplace transformation when all initial conditions are neglected.

$$F - k_1(X_1 - X_2) - c_1(X_1 s - X_2 s) = m_1 X_1 s^2$$

$$k_1(X_1 - X_2) + c_1(X_1 s - X_2 s) - k_2 X_2 - c_2 X_2 s = m_2 X_2 s^2$$

$$X_1 = \frac{F + (c_1 s + k_1)X_2}{m_1 s^2 + c_1 s + k_1}$$

$$X_1 = \frac{\left(m_2 s^2 + (c_1 + c_2)s + (k_1 + k_2)\right)X_2}{c_1 s + k_1}$$

Equate $X_1 = X_1$, then rearrange the equation,

$$G = \frac{X_2}{F}$$

$$= \frac{c_1 s + k_1}{m_1 m_2 s^4 + (m_1(c_1 + c_2) + m_2 c_1)s^3 + (m_1(k_1 + k_2) + m_2 k_1 + c_1 c_2)s^2 + (c_1 k_2 + c_2 k_1)s + k_1 k_2}$$

(b) Derive that the transfer function from the external force, f, to displacement of the first mass, x_1.

(20 Points)

Solution

$$X_2 = \frac{-F + (m_1 s^2 + c_1 s + k_1)}{c_1 s + k_1} X_1$$

$$X_2 = \frac{c_1 s + k_1}{m_2 s^2 + (c_1 + c_2)s + (k_1 + k_2)} X_1$$

Equate $X_2 = X_2$, then rearrange the equation,

$$\frac{X_1}{F} = \frac{m_2 s^2 + (c_1 + c_2)s + (k_1 + k_2)}{m_1 m_2 s^4 + (m_1(c_1 + c_2) + m_2 c_1)s^3 + (m_1(k_1 + k_2) + m_2 k_1 + c_1 c_2)s^2 + (c_1 k_2 + c_2 k_1)s + k_1 k_2}$$

(c) Determine the distance, x_1 and x_2, at the steady state, when a step force of 10 N is applied. (20 Points)

Solution

$$X_2 = \frac{c_1 s + k_1}{m_1 m_2 s^4 + (m_1(c_1 + c_2) + m_2 c_1)s^3 + (m_1(k_1 + k_2) + m_2 k_1 + c_1 c_2)s^2 + (c_1 k_2 + c_2 k_1)s + k_1 k_2} F$$

Substitute all the parameters,

$$X_2 = \frac{20s + 2000}{s^4 + 52s^3 + 5330s^2 + 18000s + 1000000} \times \frac{10}{s}$$

Apply final value theorem,

$$x_{2\_ss} = \lim_{s \to 0} s \times \frac{20s + 2000}{s^4 + 52s^3 + 5330s^2 + 18000s + 1000000} \times \frac{10}{s} = 0.02\ m$$

$$X_1 = \frac{m_2 s^2 + (c_1 + c_2)s + (k_1 + k_2)}{m_1 m_2 s^4 + (m_1(c_1 + c_2) + m_2 c_1)s^3 + (m_1(k_1 + k_2) + m_2 k_1 + c_1 c_2)s^2 + (c_1 k_2 + c_2 k_1)s + k_1 k_2} F$$

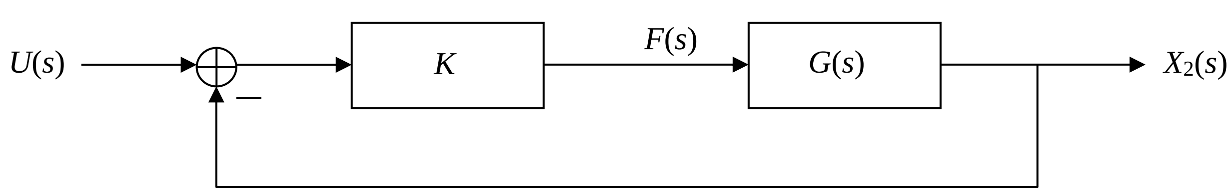
Substitute all the parameters,

$$X_1 = \frac{2s^2 + 24s + 2500}{s^4 + 52s^3 + 5330s^2 + 18000s + 1000000} \times \frac{10}{s}$$

Apply final value theorem,

$$x_{2\_ss} = \lim_{s \to 0} s \times \frac{2s^2 + 24s + 2500}{s^4 + 52s^3 + 5330s^2 + 18000s + 1000000} \times \frac{10}{s} = 0.025\ m$$

(d) In order to control position of the second mass to the desired position, a proportional controller, *K*, is applied as shown in the below figure, determine the closed loop transfer function. (10 Points)



Solution

Closed loop transfer function,

$$G_c(s) = \frac{KG}{1 + KG}$$

$$G_c(s) = \frac{20Ks + 2000K}{s^4 + 52s^3 + 5330s^2 + (18000 + 20K)s + 1000000 + 2000K}$$

(e) Determine range of the proportional controller in (d) that makes the system asymptotically stable.

(20 Points)

Solution

Characteristic equation is expressed by

$$s^4 + 52s^3 + 5330s^2 + (18000 + 20K)s + 1000000 + 2000K = 0$$

Apply Routh-Hurwitz stability criteria,

	1	5330	$1000000 + 2000K$
	52	$18000 + 20K$	0
$\frac{1}{52}$	$\frac{259160-20K}{52}$	$1000000 + 2000K$	0
$\frac{2704}{259160-20K}$	$\frac{-400K^2-584800K+1960900000}{259160-20K}$	0	0
$\frac{(259160-20K)^2}{52(-400K^2-584800K+1960900000)}$	$1000000 + 2000K$	0	0
$\frac{-400K^2-584800K+1960900000}{(259160-20K)(1000000+2000K)}$	0	0	0

The conditions of asymptotically stable are

$$259160 - 20K > 0$$

$$K < 12958$$

$$-400K^2 - 584800K + 1960900000 > 0; 400K^2 + 584800K - 1960900000 < 0$$

$$-3062.7 < K < 1600.7$$

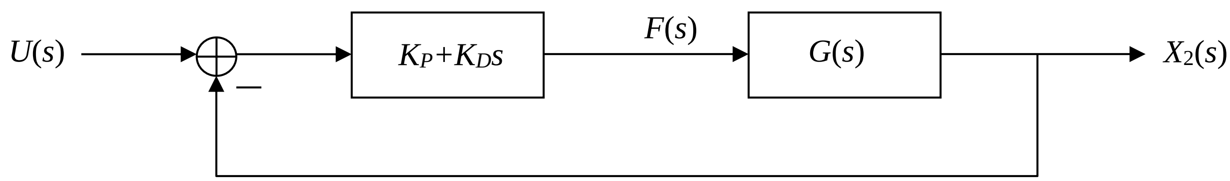
$$(259160 - 20K)(1000000 + 2000K) > 0; -(259160 - 20K)(1000000 + 2000K) < 0$$

$$-500 < K < 12958$$

Intersecting all the 3 ranges results in

$$-500 < K < 1600.7$$

(f) The poles of the uncontrolled system locate at $-25.21 \pm 66.46j, -0.79 \pm 14.05j$. The dominant time constant is 1.27 s which is too long. In order to shorten the dominant time constant, a proportional-derivative controller, PD, is applied as shown in the below figure, determine the gains that reduce the dominant time constant to 0.25 s with dominant frequency at 25 rad/s. Then determine the remaining two poles. (20 Points)



Solution

Closed loop transfer function,

$$G_c(s) = \frac{KG}{1 + KG}$$

$$G_c(s) = \frac{20K_D s^2 + (20K_P + 2000K_D)s + 2000K_P}{s^4 + 52s^3 + (5330 + 20K_D)s^2 + (18000 + 20K_P + 2000K_D)s + 1000000 + 2000K_P}$$

Thus, characteristic equation is

$$s^4 + 52s^3 + (5330 + 20K_D)s^2 + (18000 + 20K_P + 2000K_D)s + 1000000 + 2000K_P = 0$$

The desired characteristic equation is

$$(s + 4 + 25j)(s + 4 - 25j)(s + a)(s + b) = (s^2 + 8s + 641)(s^2 + (a + b)s + ab)$$

$$= s^4 + (a + b + 8)s^3 + (ab + 8a + 8b + 641)s^2 + (8ab + 641a + 641b)s + 641ab = 0$$

Thus,

$$K_P = 978.26$$

$$K_D = 13.77$$

$$a = 22 + 64.25j$$

$$b = 22 - 64.25j$$