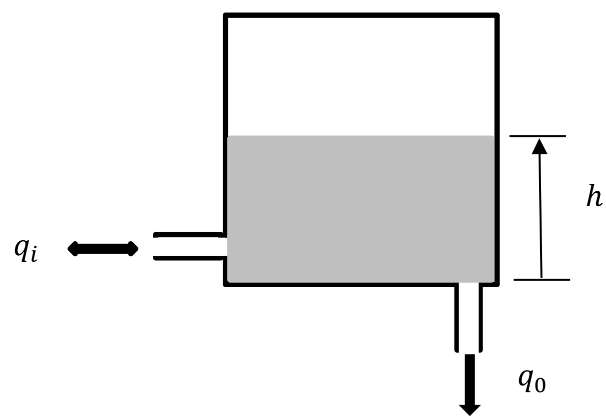
Final Examination **Control Theory AT74.10** **November 22, 2019**

Time: 9:00-11:00 hrs. Open Book
Marks: 100

Attempt all questions.

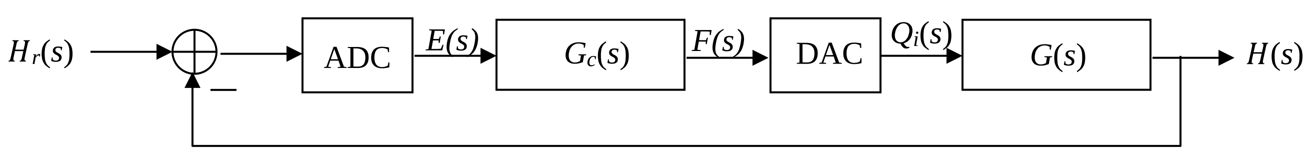
Consider the system given in the midterm examination again. Water level, h, in a cylindrical tank depends on the input flow rate, q_i , and the leakage flow rate, q_0, at the tank bottom which is proportional to the water pressure at the tank bottom as expressed by $q_0 = k\rho g h$ as shown in the figure below.



After substitution of the parameters, the transfer function from the input flow rate to the water level is expressed by

$$G = \frac{H}{Q_i} = \frac{1}{78.4s + 19.6}$$

(a) If a digital controller, $G_c(s)$, as shown in the block diagram below is used to control the water level, design the controller by direct design method when the desired close-loop transfer function, T_c, is represented in s domain by $\frac{1-e^{-Ts}}{s} \cdot \frac{0.8}{s+0.8}$. The sampling time, T, is 0.1 second. Then determine the control signal at step k, $f(k)$, as a function of control signal and error, e, at the current and previous steps. (30)



Solution

Determine the required closed loop transfer function with zero-order hold circuit,

$$T_{c2}(s) = \frac{T_c}{s} = \frac{1}{s} \cdot \frac{0.8}{s + 0.8} = \frac{1}{s} - \frac{1}{s + 0.8}$$

$$T_{c2}(z) = \frac{z}{z-1} - \frac{z}{z - e^{-0.08}} = \frac{z}{z-1} - \frac{z}{z - 0.9231}$$

$$T_c(z)=\frac{z-1}{z}T_{c2}(z)=1-\frac{z-1}{z-0.9231}=\frac{0.0769}{z-0.9231}$$

Determine the plant transfer function with zero-order hold circuit,

$$G_2(s)=\frac{1}{s}\cdot\frac{1}{78.4s+19.6}=\frac{1}{s}\cdot\frac{0.0128}{s+0.25}=\frac{0.0512}{s}-\frac{0.0512}{s+0.25}$$

$$G_2(z)=\frac{0.0512z}{z-1}-\frac{0.0512z}{z-e^{-0.025}}=\frac{0.0512z}{z-1}-\frac{0.0512z}{z-0.9753}$$

$$G(z)=\frac{z-1}{z}G_2(z)=0.0512-\frac{0.0512(z-1)}{z-0.9753}=\frac{0.0013}{z-0.9753}$$

$$G_c=\frac{T_c}{G(1-T_c)}=\frac{\frac{0.0769}{z-0.9231}}{\left(\frac{0.0013}{z-0.9753}\right)\left(1-\frac{0.0769}{z-0.9231}\right)}$$

$$G_c=\frac{F}{E}=\frac{0.0769z-0.0750}{0.0013z-0.0013}$$

$$(0.0013z-0.0013)F=(0.0769z-0.0750)E$$

$$\left(1-\frac{1}{z}\right)F=\left(59.1538-\frac{57.6923}{z}\right)E$$

$$f(k)=f(k-1)+59.1538e(k)-57.6923e(k-1)$$

(b) Determine a state-space representation of the system in the Jordan form when the water level can be obtained by measurement. Use input matrix $B=[1]$. (10)

Solution

The transfer function can be rewritten as,

$$G=\frac{H}{Q_i}=\frac{1}{78.4s+19.6}=\frac{0.0128}{s+0.25} \quad (1)$$

Thus,

$$[\dot{x}_1]=[-0.25][x_1]+[1][q_i] \quad (2)$$

$$[h]=[0.0128][x_1] \quad (3)$$

(c) From the state-space system in (b), determine the Kalman Filter gain by assuming that power spectrum of the white noise disturbance in the state dynamic, V, is 0.09 and power spectrum of the white noise in the output, W, is 0.0001. Use $F=[1]$ in the state dynamics. Then determine characteristic equation of the Kalman Filter. (25)

Solution

$$[\dot{x}_1]=[-0.25][x_1]+[1][q_i]+[1][d] \quad (1)$$

$$[h]=[0.0128][x_1]+[w] \quad (2)$$

$$K=\bar{P}C^tW^{-1} \quad (3)$$

$$K=[p_1][0.0128]^t[0.0001]^{-1}=[128p_1] \quad (4)$$

When

$$0=\dot{\bar{P}}=A\bar{P}+\bar{P}A^t-KC\bar{P}+FVF^t \quad (5)$$

Substitute all the concerned matrices,

$$[0] = [-0.25][p_1] + [p_1][-0.25] - [128p_1][0.0128][p_1] + [1][0.09][1] \quad (6)$$

$$[0] = [-0.5p_1 - 1.64p_1^2 + 0.09] \quad (7)$$

Thus,

$$[p_1] = [0.1271] \quad (8)$$

$$K = [16.2688] \quad \underline{(9)}$$

Characteristic equation of the Kalman filter is determined from

$$\left|sI - \hat{A}\right| = |sI - A + KC| = |[s] - [-0.25] + [16.2688][0.0128]| \quad (10)$$

$$\left|sI - \hat{A}\right| = |s + 0.4582| = s + 0.4582 = 0 \quad \underline{(11)}$$

(d) If the water level is controlled to a constant reference height expressed by, h_r, determine the corresponding reference state. Then remodel the state-space system in (b) by taking into consideration the reference height. Use e_h to represent the height error as the new output. (10)

Solution

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d = Ae + Bu + Ex_0 \quad (1)$$

For constant reference height,

$$[h_r] = [0.0128][x_{1r}] \quad (2)$$

$$[x_{1r}] = \left[\frac{h_r}{0.0128}\right] = [78.4h_r] \quad \underline{(3)}$$

$$[e_1] = [x_1 - 78.4h_r] \quad (4)$$

$$E = A - A_r = [-0.25] - [0] = [-0.25] \quad (5)$$

Thus,

$$[\dot{e}_1] = [-0.25][e_1] + [1][q_i] + [-0.25][78.4h_r] \quad \underline{(6)}$$

$$[e_h] = [0.0128][e_1] \quad \underline{(7)}$$

(e) Determine the formula for the input flow rate of the system in (d) as a function of the water level, h, and the reference height, h_r, that minimizes the cost function expressed by

$$V = \int_0^\infty (10000e_h^2 + 4\hat{q}_i^2)dt$$

When $q_i = \hat{q}_i + \bar{q}_i$, and $\bar{q}_i$ is used to keep the height error at zero at steady state under the reference height. Then determine characteristic equation of the regulated system. (25)

Solution

$$Q = [10000 \times 0.0128^2] = [1.6384] \quad (1)$$

$$R = [4] \quad (2)$$

Determine the gain for the state error

$$G = R^{-1}B^t\bar{M}_1 \quad (3)$$

$$R^{-1}B^t\bar{M}_1 = [4]^{-1}[1]^t[m_1] \tag{4}$$

$$R^{-1}B^t\bar{M}_1 = [0.25m_1] \tag{5}$$

When

$$0 = -\dot{\bar{M}}_1 = \bar{M}_1A + A^t\bar{M}_1 - \bar{M}_1BR^{-1}B^t\bar{M}_1 + Q \tag{6}$$

Substitute all the concerned matrices,

$$[0] = [m_1][-0.25] + [-0.25][m_1] - [m_1][1][0.25m_1] + [1.6384] \tag{7}$$

$$[0] = [-0.5m_1 - 0.25m_1^2 + 1.6384] \tag{8}$$

Thus,

$$[m_1] = [1.7484] \tag{9}$$

$$R^{-1}B^t\bar{M}_1 = [0.4371] \tag{10}$$

$$G = [0.4371] \tag{11}$$

Determine the gain of the state reference,

$$G_0 = R^{-1}B^t\bar{M}_2 \tag{12}$$

$$R^{-1}B^t\bar{M}_2 = [4]^{-1}[1]^t[m_2] \tag{13}$$

$$R^{-1}B^t\bar{M}_2 = [0.25m_2] \tag{14}$$

When

$$0 = -\dot{\bar{M}}_2 = \bar{M}_1E + \bar{M}_2A_0 + (A^t - \bar{M}_1BR^{-1}B^t)\bar{M}_2 \tag{15}$$

Substitute all the concerned matrices,

$$[0] = [1.7484][-0.25] + [m_2][0] + ([-0.25] - [0.4371][1])[m_2] \tag{16}$$

$$[0] = [-0.4371] + [-0.6871][m_2] \tag{17}$$

$$[m_2] = [-0.6362] \tag{18}$$

$$R^{-1}B^t\bar{M}_2 = [-0.1590] \tag{19}$$

$$G_0 = [-0.1590] \tag{20}$$

Thus,

$$q_i = -0.4371(e_1) + 0.1590(x_{1r}) \tag{21}$$

$$q_i = -0.4371(78.4h - 78.4h_r) + 0.1590(78.4h_r) \tag{22}$$

$$q_i = -34.2686h + 46.7342h_r \tag{23}$$

Characteristic equation of the regulated system is determined from

$$|sI - A_C| = |sI - A + BG| = |[s] - [-0.25] + [1][0.4371]| \tag{24}$$

$$|sI - A_C| = |s + 0.6871| = s + 0.6871 = 0 \tag{25}$$