Midterm Examination **Control Theory AT74.10** **September 24, 2019**

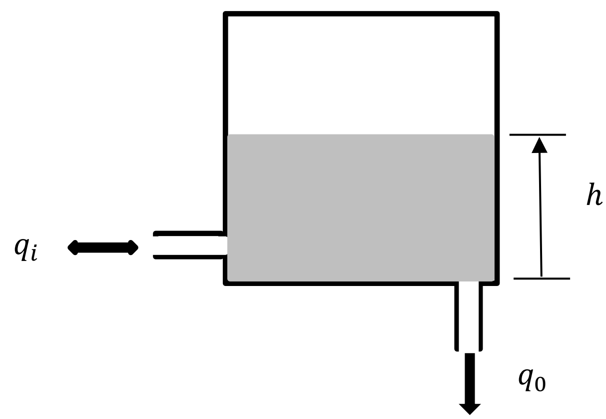
Time: 9:00-11:00 hrs. Open Book

Marks: 100

Attempt all questions.

Water level, h, in a cylindrical tank depends on the input flow rate, q_i , and the leakage flow rate, q_0, at the tank bottom as shown in the figure below. Assume the input flow can be both positive (pumping in water) and negative (pumping out water). The leakage flow rate is proportional to the water pressure at the tank bottom as expressed by $q_0 = k\rho gh$. Dynamics of the water level is expressed by

$$q_i - k\rho gh = A\dot{h}$$



When $k = 0.002\, m^3/sPa$, $\rho = 1000\, kg/m^3$, $g = 9.8\, m/s^2$, $A = 78.4\, m^2$.

1. (a) Determine the water level as a function of time, $h(t)$, if a constant input flow is $q_i = 3\, m^3/s$, and the initial water level $h(0)$, is 0.5 m. (b) Roughly plot graph of the water level as a function of time. (c) Determine the water level at steady state. (10+5+5 Points)

Solution

(a)

$$q_i - k\rho gh = A\dot{h}$$

Take Laplace transformation and consider the initial condition.

$$Q_i(s) = (As + k\rho g)H(s) - Ah(0)$$

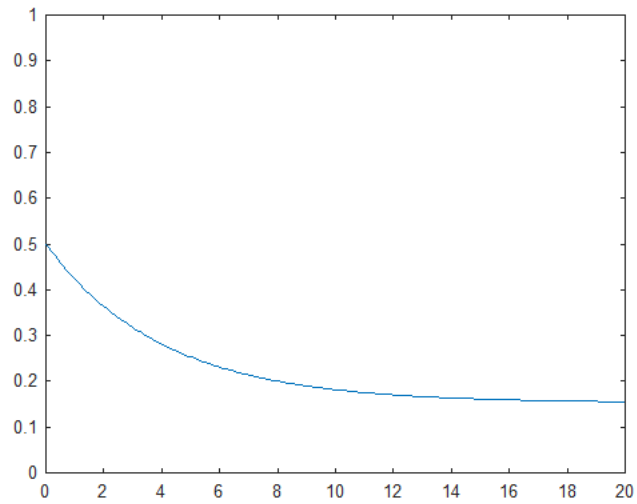
$$H(s) = \frac{Q_i(s) + Ah(0)}{As + k\rho g}$$

$$H(s) = \frac{\frac{3}{s} + 78.4 \times 0.5}{78.4s + 0.002 \times 1000 \times 9.8} = \frac{3 + 39.2s}{s(78.4s + 19.6)} = \frac{0.0383 + 0.5s}{s(s + 0.25)}$$

$$H(s) = \frac{0.0383 + 0.5s}{s(s + 0.25)} = \frac{0.1532}{s} + \frac{0.3468}{s + 0.25}$$

$$h(t) = 0.1532 + 0.3468e^{-0.25t}$$

(b)

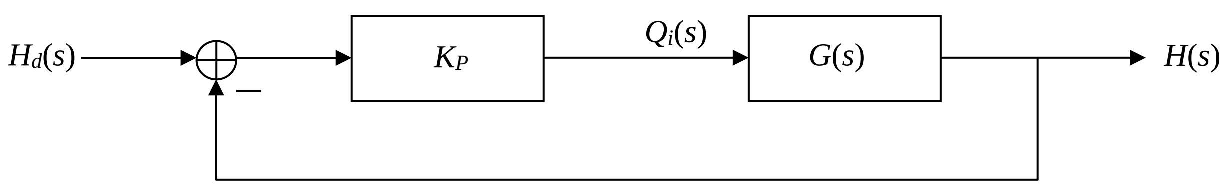


(c)

$$h(\infty) = 0.1532 + 0.3468e^{-0.25\infty} = 0.1532\ m$$

2. If the water level has to be controlled to a desired level, H_d, a proportional controller with gain, K_P, is applied. (a) Determine close-loop transfer function. (b) Plot root-locus diagram. (c) Plot Nyquist diagram. (d) Plot bode diagram of the loop transfer function. (e) If the maximum proportional gain is limited to 100, determine the minimum percentage of steady-state error and (f) the shortest dominant time constant.

(7+7+7+7+6+6 Points)



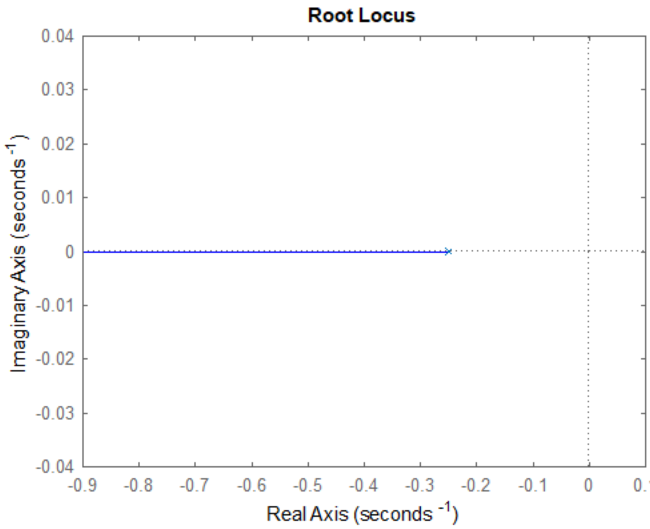
Solution

(a)

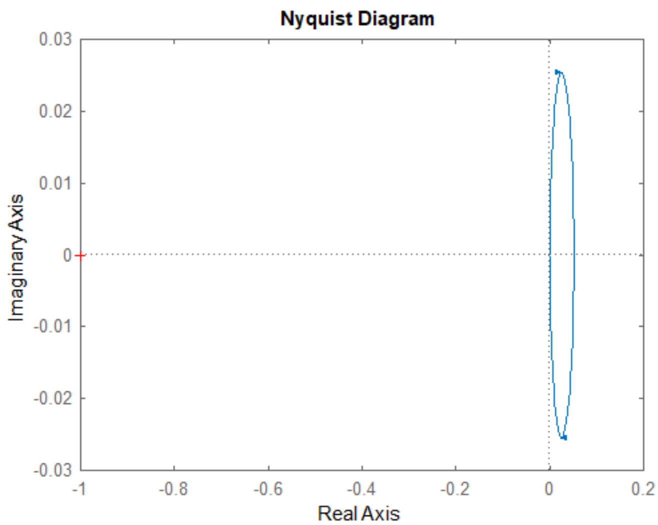
$$G = \frac{H}{Q_i} = \frac{1}{As + k\rho g} = \frac{1}{78.4s + 0.002 \times 1000 \times 9.8} = \frac{1}{78.4s + 19.6}$$

$$\frac{H(s)}{H_d(s)} = \frac{K_P G}{1 + K_P G} = \frac{K_P}{78.4s + 19.6 + K_P}$$

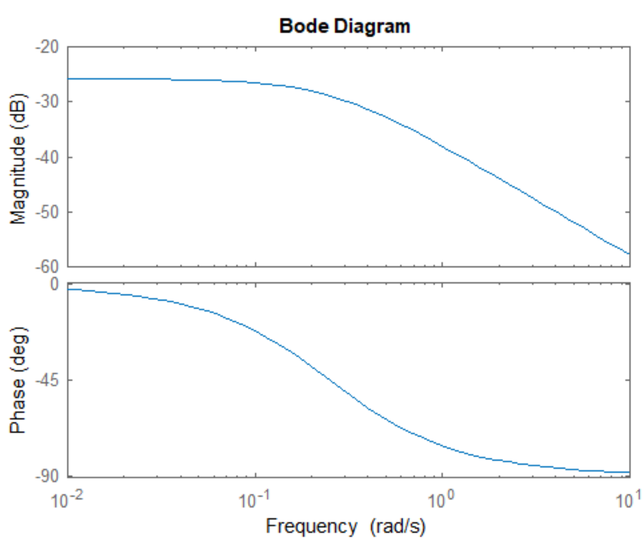
(b)


Root Locus
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
Imaginary Axis (seconds -1)
-0.9 -0.8 -0.7 -0.6 -0.5 -0.4 -0.3 -0.2 -0.1 0 0.1
Real Axis (seconds -1)

(c)


Nyquist Diagram
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
Imaginary Axis
-1 -0.8 -0.6 -0.4 -0.2 0 0.2
Real Axis

(d)


Bode Diagram
-20
-30
-40
-50
-60
Magnitude (dB)
0
-45
-90
Phase (deg)
10-2 10-1 100 101
Frequency (rad/s)

(e) If select $K_P = 100$,

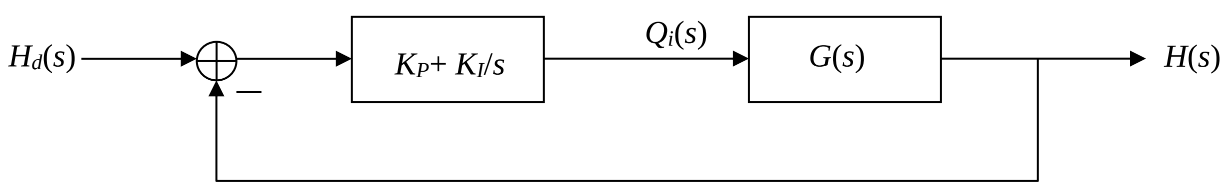
$$\frac{E(s)}{H_d(s)} = \frac{1}{1+K_P G} = \frac{1}{78.4s + 19.6 + 100} = \frac{1}{78.4s + 119.6}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = s\frac{1}{78.4s + 119.6}\frac{1}{s} = 0.84\%$$

(f)

$$\tau = \frac{78.4}{119.6} = 0.66\ s$$

3. If steady-state error of the water level is unacceptable a proportional-integral controller $(K_P + {}^{K_I}/_{s})$, is applied. (a) Determine the close-loop transfer function. (b) If the maximum proportional gain is limited to 100, determine the shortest dominant time constant, and (c) the range of the integral gain that achieves the shortest dominant time constant. (d) If the initial water level is 0.5 m, the initial rate of water level is 0 m/s, the desired water level is a step function of 1 m, determine the water level as a function of time when the proportional gain is selected at 100 and the integral gain is selected at the value that results in critical damp response, then (e) roughly plot graph of the response as a function of time. (8+8+8+8+8 Points)



Solution

(a)

$$G = \frac{H}{Q_i} = \frac{1}{As + k\rho g} = \frac{1}{78.4s + 0.002 \times 1000 \times 9.8} = \frac{1}{78.4s + 19.6}$$

$$\frac{H(s)}{H_d(s)} = \frac{\left(K_P + {}^{K_I}/_{s}\right)G}{1 + \left(K_P + {}^{K_I}/_{s}\right)G} = \frac{K_P s + K_I}{78.4s^2 + (19.6 + K_P)s + K_I}$$

(b)

From the characteristic equation,

$$78.4s^2 + (19.6 + K_P)s + K_I = 0$$

If select $K_P = 100$,

$$78.4s^2 + 119.6s + K_I = 0$$

The roots of the characteristic equation are determined from

$$s = \frac{-119.6 \pm \sqrt{119.6^2 - 4 \times 78.4K_I}}{2 \times 78.4}$$

The shortest time constant achieves during critical damp and under damp,

$$\tau = \frac{2 \times 78.4}{119.6} = 1.31\ s$$

(c)

$$119.6^2 - 4 \times 78.4K_I \le 0$$

$$K_I \ge 45.61$$

(d)

Critical damp in the second order system occurs when the roots are repeated. If $K_P = 100,\ K_I = 45.61$.

From the system transfer function,

$$\frac{H(s)}{H_d(s)} = \frac{K_P s + K_I}{78.4s^2 + (19.6 + K_P)s + K_I} = \frac{100s + 45.61}{78.4s^2 + 119.6s + 45.61}$$

$$(78.4s^2 + 119.6s + 45.61)H(s) = (100s + 45.61)H_d(s)$$

If consider the initial conditions,

$$(78.4s^2 + 119.6s + 45.61)H(s) - 78.4sh(0) - 78.4\dot{h}(0) - 119.6h(0)$$

$$= (100s + 45.61)H_d(s) - 100h_d(0)$$

Substitute all the initial conditions,

$$(78.4s^2 + 119.6s + 45.61)H(s) - 78.4s(0.5) - 78.4(0) - 119.6(0.5) = (100s + 45.61)\frac{1}{s} - 100(0)$$

$$(78.4s^2 + 119.6s + 45.61)H(s) - 78.4s(0.5) - 119.6(0.5) = (100s + 45.61)\frac{1}{s}$$

$$H(s) = \frac{39.2s^2 + 159.8s + 45.61}{s(78.4s^2 + 119.6s + 45.61)} = \frac{0.5s^2 + 2.0383s + 0.5818}{s(s^2 + 1.5255s + 0.5818)}$$

$$H(s) = \frac{A}{s} + \frac{B}{s + 0.7628} + \frac{C}{(s + 0.7628)^2} = \frac{(A + B)s^2 + (1.5255A + 0.7628B + C)s + 0.5818A}{s(s^2 + 1.5255s + 0.5818)}$$

$$A + B = 0.5$$

$$1.5255A + 0.7628B + C = 2.0383$$

$$0.5818A = 0.5818$$

$$H(s) = \frac{1}{s} - \frac{0.5}{s + 0.7628} + \frac{0.8942}{(s + 0.7628)^2}$$

$$h(t) = 1 - 0.5e^{-0.7628t} + 0.8942te^{-0.7628t}$$

(e)