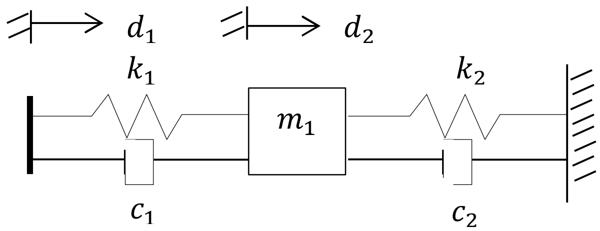
Final Examination **Control Theory AT74.10** **December 1, 2020**

Time: 13:00-15:00 hrs. Open Book

Marks: 100

Attempt all questions.

Consider the system given in the midterm examination again. An input motion, d_1, is applied to a 1-mass-2-spring-2-damper system as shown in the below figure. d_1 and d_2 are the distances of the input motion and the mass, m_1, from the fixed points. Both distances are 0 at the steady state when there is no input motion. Use $m_1 = 1000$ kg, $k_1 = 2000$ N/m, $k_2 = 3000$ N/m, $c_1 = 1000$ Ns/m, $c_2 = 5000$ Ns/m.



The transfer function from the input motion, d_1, to the output motion of the mass, d_2, is $G = \frac{d_2}{d_1} = \frac{c_1 s + k_1}{m_1 s^2 + (c_1 + c_2)s + k_1 + k_2}$.

(a) Determine a state-space representation of the system in the first-companion form when the state variables are numbered from right to left with the input, u, of the distance, d_1, and the output, y, of the distance, d_2. (15)

<u>Solution</u>

Transfer function from the input of the distance, d_1, and the output, y, of the distance, d_2,

$$G = \frac{d_2}{d_1} = \frac{c_1 s + k_1}{m_1 s^2 + (c_1 + c_2)s + k_1 + k_2} \tag{1}$$

Substitute all the parameters,

$$\frac{d_2}{d_1} = \frac{1000s + 2000}{1000s^2 + 6000s + 5000} = \frac{s+2}{s^2 + 6s + 5} \tag{2}$$

Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u] \tag{3}$$

$$[y] = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4}$$

(b) If the output is controlled to a constant reference distance, d_{2r}, remodel the state-space system in (a) by taking into consideration the reference. (15)

Solution

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d = Ae + Bu + Ex_0 \quad (1)$$

For constant reference distance of the first-companion from,

$$[d_{2r}] = [2 \quad 1]\begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} = [2 \quad 1]\begin{bmatrix} x_{1r} \\ 0 \end{bmatrix} \quad (2)$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{1r} \\ x_2 - x_{2r} \end{bmatrix} = \begin{bmatrix} x_1 - x_{1r} \\ x_2 \end{bmatrix} \quad (3)$$

$$E = A - A_r = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \quad (4)$$

Thus,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[u] + \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}\begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} \quad \underline{(5)}$$

$$[y_e] = [2 \quad 1]\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \underline{(6)}$$

(c) Determine the formula for the input of the distance, d_1, that minimizes the cost function expressed by

$$V = \int_0^{\infty} \left((d_2 - d_{2r})^2 + \hat{d}_1^2\right)dt$$

When $d_1 = \hat{d}_1 + \bar{d}_1$, and $\bar{d}_1$ is used to keep the distance error, $(d_2 - d_{2r})$, at zero at steady state under the distance reference, d_{2r}. Then determine characteristic equation of the regulated system. (35)

Solution

$$Q = C^T C = \begin{bmatrix} 2 \\ 1 \end{bmatrix}[2 \quad 1] = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad (1)$$

$$R = [1] \quad (2)$$

Determine the gain for the state error

$$G = R^{-1}B^t\bar{M}_1 \quad (3)$$

$$R^{-1}B^t\bar{M}_1 = [1]^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}^t\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \quad (4)$$

$$R^{-1}B^t\bar{M}_1 = [m_2 \quad m_3] \quad (5)$$

When

$$0 = -\dot{\bar{M}}_1 = \bar{M}_1 A + A^t\bar{M}_1 - \bar{M}_1 B R^{-1} B^t \bar{M}_1 + Q \quad (6)$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 & -5 \\ 1 & -6 \end{bmatrix}\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} - \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[m_2 \quad m_3] + \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad (7)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -m_2^2 - 10m_2 + 4 & m_1 - 6m_2 - 5m_3 - m_2m_3 + 2 \\ m_1 - 6m_2 - 5m_3 - m_2m_3 + 2 & 2m_2 - 12m_3 - m_3^2 + 1 \end{bmatrix} \tag{8}$$

Thus,

$$\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = \begin{bmatrix} 1.10 & 0.39 \\ 0.39 & 0.15 \end{bmatrix} \tag{9}$$

$$G = R^{-1}B^t\bar{M}_1 = [0.39 \quad 0.15] \tag{10}$$

$$A_c = A - BG = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[0.39 \quad 0.15] = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0.39 & 0.15 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5.39 & -6.15 \end{bmatrix} \tag{11}$$

Determine the gain of the state reference,

$$G_0 = R^{-1}B^t\bar{M}_2 \tag{12}$$

$$R^{-1}B^t\bar{M}_2 = [1]^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}^t \begin{bmatrix} m_{21} & m_{22} \\ m_{23} & m_{24} \end{bmatrix} \tag{13}$$

$$R^{-1}B^t\bar{M}_2 = [m_{23} \quad m_{24}] \tag{14}$$

When

$$0 = -\dot{\bar{M}}_2 = \bar{M}_1E + \bar{M}_2A_0 + (A^t - \bar{M}_1BR^{-1}B^t)\bar{M}_2 \tag{15}$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1.10 & 0.39 \\ 0.39 & 0.15 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} + \begin{bmatrix} m_{21} & m_{22} \\ m_{23} & m_{24} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -5.39 \\ 1 & -6.15 \end{bmatrix}\begin{bmatrix} m_{21} & m_{22} \\ m_{23} & m_{24} \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1.93 & -1.22 \\ -0.73 & -0.490 \end{bmatrix} + \begin{bmatrix} -5.39m_{23} & -5.39m_{24} \\ m_{21} - 6.15m_{23} & m_{22} - 6.15m_{24} \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} m_{21} & m_{22} \\ m_{23} & m_{24} \end{bmatrix} = \begin{bmatrix} -1.47 & -0.90 \\ -0.36 & -0.23 \end{bmatrix} \tag{18}$$

$$G_0 = R^{-1}B^t\bar{M}_2 = [-0.36 \quad -0.23] \tag{19}$$

Thus,

$$u = d_1 = -[0.39 \quad 0.15]\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - [-0.36 \quad -0.23]\begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} \tag{20}$$

Characteristic equation of the regulated system is determined from

$$|sI - A_C| = \begin{vmatrix} s & -1 \\ 5.39 & s + 6.15 \end{vmatrix} \tag{21}$$

$$|sI - A_C| = s^2 + 6.15s + 5.39 = 0 \tag{22}$$

(d) Determine the Kalman Filter gain by assuming that power spectrum of the white noise disturbance in the state dynamic, V, is [1] and power spectrum of the white noise in the output, W, is [1]. Use $F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in the state dynamics. Then determine characteristic equation of the Kalman Filter. (35)

Solution

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[u] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[v] \tag{1}$$

$$[y] = [2 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [w] \tag{2}$$

$$K = \bar{P}C^t W^{-1} \tag{3}$$

$$K = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} [2 \quad 1]^t [1]^{-1} = \begin{bmatrix} 2p_1 + p_2 \\ 2p_2 + p_3 \end{bmatrix} \tag{4}$$

When

$$0 = \dot{\bar{P}} = A\bar{P} + \bar{P}A^t - KC\bar{P} + FVF^t \tag{5}$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & -5 \\ 1 & -6 \end{bmatrix} - \begin{bmatrix} 2p_1 + p_2 \\ 2p_2 + p_3 \end{bmatrix} [2 \quad 1] \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1][0 \quad 1] \tag{6}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2p_2 - 4p_1^2 - 4p_1p_2 - p_2^2 & -5p_1 - 6p_2 + p_3 - 4p_1p_2 - 2p_1p_3 - 2p_2^2 - p_2p_3 \\ -5p_1 - 6p_2 + p_3 - 4p_1p_2 - 2p_1p_3 - 2p_2^2 - p_2p_3 & -10p_2 - 12p_3 - 4p_2^2 - 4p_2p_3 - p_3^2 + 1 \end{bmatrix} \tag{7}$$

Thus,

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0.015 & 0.00 \\ 0.00 & 0.08 \end{bmatrix} \tag{8}$$

$$K = \begin{bmatrix} 0.03 \\ 0.08 \end{bmatrix} \tag{9}$$

Characteristic equation of the Kalman filter is determined from

$$\left|sI - \hat{A}\right| = |sI - A + KC| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} + \begin{bmatrix} 0.03 \\ 0.08 \end{bmatrix} [2 \quad 1] \right| \tag{10}$$

$$\left|sI - \hat{A}\right| = \begin{vmatrix} s + 0.06 & -0.97 \\ 5.16 & s + 6.08 \end{vmatrix} = s^2 + 6.14s + 5.37 = 0 \tag{11}$$