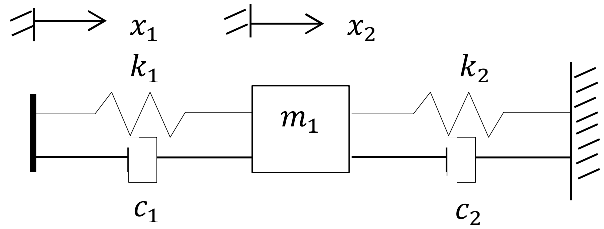
Midterm Examination **Control Theory AT74.10** **September 29, 2020**

Time: 9:00-11:00 hrs. Open Book
Marks: 100

Attempt all questions.

An input motion, x_1, is applied to a 1-mass-2-spring-2-damper system as shown in the below figure. x_1 and x_2 are the distances of the input motion and the mass, m_1, from the fixed points. Both distances are 0 at the steady state when there is no input motion. Use $m_1 = 1000$ kg, $k_1 = 2000$ N/m, $k_2 = 3000$ N/m, $c_1 = 1000$ Ns/m, $c_2 = 5000$ Ns/m.



(a) Derive that the transfer function from the input motion, x_1, to the output motion of the mass, x_2, is $G = \frac{X_2}{X_1} = \frac{c_1 s + k_1}{m_1 s^2 + (c_1 + c_2)s + k_1 + k_2}$. (10 Points)

Solution

$$k_1(x_1 - x_2) + c_1(\dot{x}_1 - \dot{x}_2) - k_2 x_2 - c_2 \dot{x}_2 = m_1 \ddot{x}_2$$

Take Laplace transformation when all initial conditions are neglected.

$$k_1(X_1 - X_2) + c_1(X_1 s - X_2 s) - k_2 X_2 - c_2 X_2 s = m_1 X_2 s^2$$

$$(c_1 s + k_1)X_1 = (m_1 s^2 + (c_1 + c_2)s + k_1 + k_2)X_2$$

$$\frac{X_2}{X_1} = \frac{c_1 s + k_1}{m_1 s^2 + (c_1 + c_2)s + k_1 + k_2}$$

(b) Determine the distance, $x_2(t)$, when the input motion is expressed by $x_1(t) = 2sin(6t)$. (20 Points)

Solution

$$X_2 = \frac{c_1 s + k_1}{m_1 s^2 + (c_1 + c_2)s + k_1 + k_2} X_1$$

Substitute all the parameters,

$$X_2 = \frac{1000s + 2000}{1000s^2 + 6000s + 5000} \times \frac{12}{s^2 + 36} = \frac{s + 2}{s^2 + 6s + 5} \times \frac{12}{s^2 + 36}$$

$$X_2 = \frac{s + 2}{s^2 + 6s + 5} \times \frac{12}{s^2 + 36} = \frac{A}{s + 1} + \frac{B}{s + 5} + \frac{Cs + D}{s^2 + 36}$$

$$A(s^3+5s^2+36s+180)+B(s^3+s^2+36s+36)+C(s^3+6s^2+5s)+D(s^2+6s+5)=12s+24$$

$$A+B+C=0$$

$$5A+B+6C+D=0$$

$$36A+36B+5C+6D=12$$

$$180A+36B+5D=24$$

$$A=0.08, B=0.15, C=-0.23, D=0.82$$

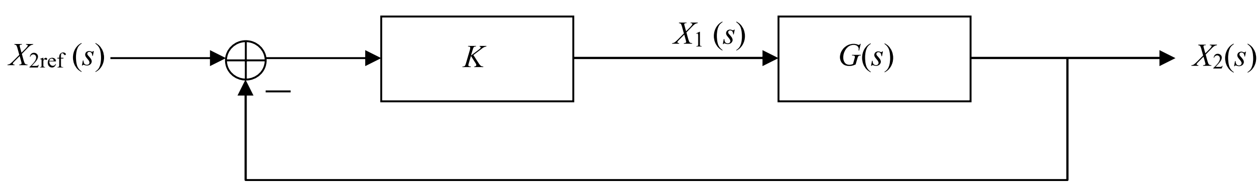
$$X_2=\frac{0.08}{s+1}+\frac{0.15}{s+5}+\frac{-0.23s+0.82}{s^2+36}$$

$$x_2(t)=0.08e^{-t}+0.15e^{-5t}-0.23cos(6t)+0.14sin(6t)$$

$$x_2(t)=0.08e^{-t}+0.15e^{-5t}+0.27sin(6t-\theta)$$

$$\theta=atan(1.64)=1.02\,rad=58.67°$$

(c) In order to control position of the mass to the desired position, a proportional controller, K, is applied as shown in the below figure, determine the closed loop transfer function, the steady-state response as a function of proportional gain when the reference is a unit step function, and the shortest dominant time constant and the gain that provides this shortest dominant time constant. (20 Points)



Solution

Closed loop transfer function,

$$G_c(s)=\frac{KG}{1+KG}$$

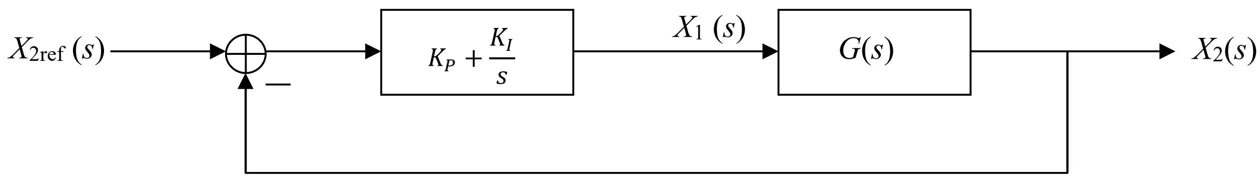
$$G_c(s)=\frac{Ks+2K}{s^2+(6+K)s+5+2K}$$

Steady-state response is determined from final value theorem,

$$x_{2ss}=\lim_{s\to 0} s\times\frac{Ks+2K}{s^2+(6+K)s+5+2K}\times\frac{1}{s}=\frac{2K}{5+2K}$$

Since there are 2 poles at -1 and -5 and 1 zero at -2, the shortest dominant time constant is obtained when the pole of the closed loop system is at the zero of the loop transfer function ($s=-2$), when the gain is infinity. The shortest dominant time constant is $\frac{1}{2}=0.5$ second.

(d) In order to improve further from the proportional controller in (c), a proportional-integral controller, $K_P + \frac{K_I}{s}$, is applied as shown in the below figure, determine the closed loop transfer function, the steady-state response when the reference is a unit step function, and the proportional gain and integral gain and the remaining pole when two poles are real having equal time constant of 0.25 second. (20 Points)



Solution

Closed loop transfer function,

$$G_c(s) = \frac{\left(K_P + \frac{K_I}{s}\right)G}{1 + \left(K_P + \frac{K_I}{s}\right)G}$$

$$G_c(s) = \frac{K_p s^2 + (2K_p + K_I)s + 2K_I}{s^3 + (6 + K_p)s^2 + (2K_p + K_I + 5)s + 2K_I}$$

Steady-state response is determined from final value theorem,

$$x_{2ss} = \lim_{s \to 0} s \times \frac{K_p s^2 + (2K_p + K_I)s + 2K_I}{s^3 + (6 + K_p)s^2 + (2K_p + K_I + 5)s + 2K_I} \times \frac{1}{s} = 1$$

The desired characteristic equation is expressed by

$$(s + 4)(s + 4)(s + a) = s^3 + (8 + a)s^2 + (16 + 8a)s + 16a = 0$$

Equate characteristic equation with the desired characteristic equation,

$$s^3 + (6 + K_p)s^2 + (2K_p + K_I + 5)s + 2K_I = s^3 + (8 + a)s^2 + (16 + 8a)s + 16a = 0$$

$$6 + K_p = 8 + a$$

$$2K_p + K_I + 5 = 16 + 8a$$

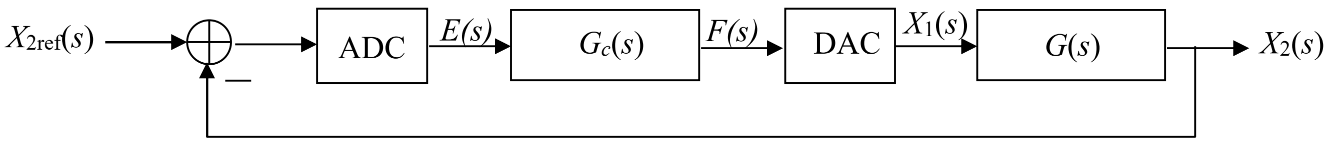
$$2K_I = 16a$$

$$K_p = 5.5$$

$$K_I = 28$$

$$a = 3.5$$

(e) If a digital controller, $G_c(s)$, as shown in the block diagram below is used to control position of the mass, design the controller by direct design method when the desired closed loop transfer function, T_c, is represented in s domain by $\frac{1-e^{-Ts}}{s}\cdot\frac{4}{s+4}$. The sampling time, T, is 0.1 second. Then determine the control signal at step k, $f(k)$, as a function of control signal and error, e, at the current and previous steps. (30 Points)



Solution

Determine the required closed loop transfer function with zero-order hold circuit,

$$T_{c2}(s) = \frac{T_c}{s} = \frac{1}{s}\cdot\frac{4}{s+4} = \frac{1}{s} - \frac{1}{s+4}$$

$$T_{c2}(z) = \frac{z}{z-1} - \frac{z}{z-e^{-0.4}}$$

$$T_{c2}(z) = \frac{z}{z-1} - \frac{z}{z-0.67}$$

$$T_c(z) = \frac{z-1}{z}T_{c2}(z) = 1 - \frac{z-1}{z-0.67} = \frac{0.33}{z-0.67}$$

Determine the plant transfer function with zero-order hold circuit,

$$G_2(s) = \frac{1}{s}\cdot\frac{s+2}{s^2+6s+5} = \frac{0.4}{s} - \frac{0.25}{s+1} - \frac{0.15}{s+5}$$

$$G_2(z) = \frac{0.4z}{z-1} - \frac{0.25z}{z-e^{-0.1}} - \frac{0.15z}{z-e^{-0.5}} = \frac{0.4z}{z-1} - \frac{0.25z}{z-0.90} - \frac{0.15z}{z-0.61}$$

$$G(z) = \frac{z-1}{z}G_2(z) = 0.4 - \frac{0.25(z-1)}{z-0.90} - \frac{0.15(z-1)}{z-0.61} = \frac{0.08z-0.07}{z^2-1.51z+0.55}$$

$$G_c = \frac{T_c}{G(1-T_c)} = \frac{\frac{0.33}{z-0.67}}{\left(\frac{0.08z-0.07}{z^2-1.51z+0.55}\right)\left(1-\frac{0.33}{z-0.67}\right)}$$

$$G_c = \frac{F}{E} = \frac{0.33z^2-0.50z+0.18}{0.08z^2-0.15z+0.07}$$

$$(0.08z^2-0.15z+0.07)F = (0.33z^2-0.50z+0.18)E$$

$$\left(1-\frac{1.88}{z}+\frac{0.88}{z^2}\right)F = \left(4.13-\frac{6.25}{z}+\frac{2.25}{z^2}\right)E$$

$$f(k) = 1.88f(k-1) - 0.88f(k-2) + 4.13e(k) - 6.25e(k-1) + 2.25e(k-2)$$