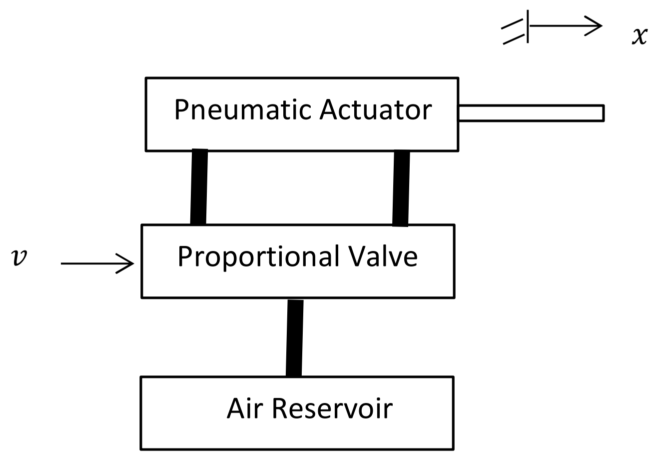
**Final Examination** **Control Theory AT74.10** **7 December, 2021**

Time: 9:00-11:00 hrs. Open Book
Marks: 100

Attempt all questions.

Consider the system given in the midterm examination again. Input voltage of a proportional valve, $v$ (volt), is used to control position of a pneumatic actuator, $x$ (meter), as shown in the below figure and expressed by a transfer function, $G = \frac{x}{v} = \frac{4}{20s^2+10s}$.




(a) Determine a state-space representation of the system in the first-companion form when the state variables are numbered from right to left with the input, $u$, of the voltage, $v$, and the output, $y$, of the actuator position, $x$. (15)

**Solution**

Transfer function,

$$G = \frac{x}{v} = \frac{4}{20s^2+10s} = \frac{0.2}{s^2+0.5s} \quad (1)$$

Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u] \quad \underline{(2)}$$

$$[y] = \begin{bmatrix} 0.2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underline{(3)}$$

(b) If the output is controlled to a constant reference actuator position, $x_r$, remodel the state-space system in (a) by taking into consideration the reference. (15)

**Solution**

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d = Ae + Bu + Ex_0 \quad (1)$$

For constant reference position of the first-companion from,

$$[x_r] = [0.2 \quad 0]\begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} = [0.2 \quad 0]\begin{bmatrix} x_{1r} \\ 0 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{1r} \\ x_2 - x_{2r} \end{bmatrix} = \begin{bmatrix} x_1 - x_{1r} \\ x_2 \end{bmatrix} \tag{3}$$

$$E = A - A_r = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} \tag{4}$$

Thus,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[u] + \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix}\begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} \tag{5}$$

$$[y_e] = [0.2 \quad 0]\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{6}$$

(c) Determine the formula for the input of the voltage, $v$ , that minimizes the cost function expressed by

$$V = \int_0^\infty (10(x - x_r)^2 + v_1^2)dt$$

as a function of the actuator position, its derivative, and the position reference $(x, \dot{x}, x_r)$ only, when $v = v_1 + \bar{v}$, and $\bar{v}$ is used to keep the position error $(x - x_r)$, at zero at steady state under the position reference, $x_r$. Then determine characteristic equation of the regulated system. (35)

**Solution**

$$Q = C^T C = 10\begin{bmatrix} 0.2 \\ 0 \end{bmatrix}[0.2 \quad 0] = \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

$$R = [1] \tag{2}$$

Determine the gain for the state error

$$G = R^{-1}B^t\bar{M}_1 \tag{3}$$

$$R^{-1}B^t\bar{M}_1 = [1]^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}^t\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \tag{4}$$

$$R^{-1}B^t\bar{M}_1 = [m_2 \quad m_3] \tag{5}$$

When

$$0 = -\dot{\bar{M}}_1 = \bar{M}_1 A + A^t\bar{M}_1 - \bar{M}_1 B R^{-1} B^t \bar{M}_1 + Q \tag{6}$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -0.5 \end{bmatrix}\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} - \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[m_2 \quad m_3] + \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -m_2^2 + 0.4 & m_1 - 0.5m_2 - m_2 m_3 \\ m_1 - 0.5m_2 - m_2 m_3 & 2m_2 - m_3 - m_3^2 \end{bmatrix} \tag{8}$$

Thus,

$$\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = \begin{bmatrix} 0.78 & 0.63 \\ 0.63 & 0.73 \end{bmatrix} \tag{9}$$

$$G = R^{-1}B^t\bar{M}_1 = [0.63 \quad 0.73] \tag{10}$$

$$A_c = A - BG = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[0.63 \quad 0.73] = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0.63 & 0.73 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.63 & -1.23 \end{bmatrix} \tag{11}$$

Determine the gain of the state reference,

$$G_0 = R^{-1}B^t\bar{M}_2 \tag{12}$$

$$R^{-1}B^t\bar{M}_2 = [1]^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}^t \begin{bmatrix} m_{21} & m_{22} \\ m_{23} & m_{24} \end{bmatrix} \tag{13}$$

$$R^{-1}B^t\bar{M}_2 = [m_{23} \quad m_{24}] \tag{14}$$

When

$$0 = -\dot{\bar{M}}_2 = \bar{M}_1 E + \bar{M}_2 A_0 + (A^t - \bar{M}_1 B R^{-1} B^t)\bar{M}_2 \tag{15}$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.78 & 0.63 \\ 0.63 & 0.73 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} + \begin{bmatrix} m_{21} & m_{22} \\ m_{23} & m_{24} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -0.63 \\ 1 & -1.23 \end{bmatrix}\begin{bmatrix} m_{21} & m_{22} \\ m_{23} & m_{24} \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.47 \\ 0 & 0.27 \end{bmatrix} + \begin{bmatrix} -0.63m_{23} & -0.63m_{24} \\ m_{21} - 1.23m_{23} & m_{22} - 1.23m_{24} \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} m_{21} & m_{22} \\ m_{23} & m_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0.65 \\ 0 & 0.75 \end{bmatrix} \tag{18}$$

$$G_0 = R^{-1}B^t\bar{M}_2 = [0 \quad 0.75] \tag{19}$$

Thus,

$$u = v = -[0.63 \quad 0.73]\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - [0 \quad 0.75]\begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} \tag{20}$$

Since,

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{1r} \\ x_2 - x_{2r} \end{bmatrix} = \begin{bmatrix} x_1 - x_{1r} \\ x_2 \end{bmatrix} \tag{21}$$

and

$$[y] = [0.2 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{22}$$

Thus

$$\begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} = \begin{bmatrix} \frac{y_r}{0.2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{x_r}{0.2} \\ 0 \end{bmatrix} \tag{23}$$

and

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \frac{y}{0.2} - \frac{y_r}{0.2} \\ \frac{\dot{y}}{0.2} \end{bmatrix} = \begin{bmatrix} \frac{x}{0.2} - \frac{x_r}{0.2} \\ \frac{\dot{x}}{0.2} \end{bmatrix} \tag{24}$$

Finally,

$$u = v = -[0.63 \quad 0.73]\begin{bmatrix} \frac{x}{0.2} - \frac{x_r}{0.2} \\ \frac{\dot{x}}{0.2} \end{bmatrix} - [0 \quad 0.75]\begin{bmatrix} \frac{x_r}{0.2} \\ 0 \end{bmatrix} \tag{25}$$

Characteristic equation of the regulated system is determined from

$$|sI - A_C| = \begin{vmatrix} s & -1 \\ 0.63 & s + 1.23 \end{vmatrix} \quad (26)$$

$$|sI - A_C| = s^2 + 1.23s + 0.63 = 0 \quad \underline{(27)}$$

(d) Determine the Kalman Filter gain by assuming that power spectrum of the white noise disturbance in the state dynamic, $V$, is [10] and power spectrum of the white noise in the output, $W$, is [1]. Use $F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in the state dynamics. Then determine characteristic equation of the Kalman Filter. (35)

**<u>Solution</u>**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [v] \quad (1)$$

$$[y] = \begin{bmatrix} 0.2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [w] \quad (2)$$

$$K = \bar{P} C^t W^{-1} \quad (3)$$

$$K = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0.2 & 0 \end{bmatrix}^t [1]^{-1} = \begin{bmatrix} 0.2p_1 \\ 0.2p_2 \end{bmatrix} \quad (4)$$

When

$$0 = \dot{\bar{P}} = A\bar{P} + \bar{P}A^t - KC\bar{P} + FVF^t \quad (5)$$

Substitute all the concerned matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & -0.5 \end{bmatrix} - \begin{bmatrix} 0.2p_1 \\ 0.2p_2 \end{bmatrix} \begin{bmatrix} 0.2 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [10] \begin{bmatrix} 0 & 1 \end{bmatrix} \quad (6)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2p_2 - 0.04p_1^2 & -0.5p_2 + p_3 - 0.04p_1p_2 \\ -0.5p_2 + p_3 - 0.04p_1p_2 & p_3 - 0.04p_2^2 + 10 \end{bmatrix} \quad (7)$$

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 18.27 & 6.68 \\ 6.68 & 8.22 \end{bmatrix} \quad (8)$$

$$K = \begin{bmatrix} 3.65 \\ 1.34 \end{bmatrix} \quad \underline{(9)}$$

Characteristic equation of the Kalman filter is determined from

$$|sI - \hat{A}| = |sI - A + KC| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} + \begin{bmatrix} 3.65 \\ 1.34 \end{bmatrix} \begin{bmatrix} 0.2 & 0 \end{bmatrix} \right| \quad (10)$$

$$|sI - \hat{A}| = \begin{vmatrix} s + 0.73 & -1 \\ 0.27 & s + 0.5 \end{vmatrix} = s^2 + 1.23s + 0.27 = 0 \quad \underline{(11)}$$