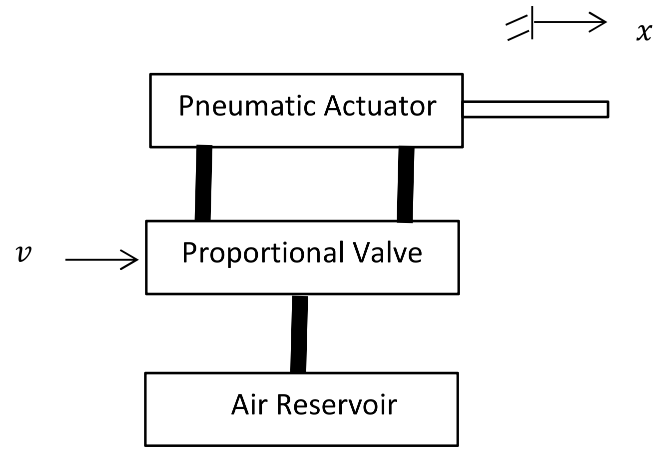
Midterm Examination **Control Theory AT74.10** **October 5, 2021**

Time: 9:00-11:00 hrs. Open Book

Marks: 100

Attempt all questions.

Input voltage of a proportional valve, v (volt), is used to control position of a pneumatic actuator, x (meter), in a system as shown in the below figure and expressed by a transfer function, $G = \frac{x}{v} = \frac{4}{20s^2+10s}$.



(a) Determine the position, $x(t)$, when the input voltage is expressed by $v(t) = 5e^{-0.1t}sin(0.2t)$ when the initial position and initial speed of the actuator are 0 m and 0 m/s respectively. (20 Points)

Solution

The input voltage is converted to Laplacian domain,

$$v(s) = \frac{5 \times 0.2}{(s+0.1)^2 + 0.04} = \frac{1}{(s+0.1)^2 + 0.04}$$

The output is the multiplication of the transfer function with the input in Laplacian domain,

$$x = Gv = \frac{4}{20s^2 + 10s} \times \frac{1}{(s+0.1)^2 + 0.04} = \frac{0.2}{s(s+0.5)((s+0.1)^2 + 0.04)}$$

$$x = \frac{A}{s} + \frac{B}{s+0.5} + \frac{Cs + D}{(s+0.1)^2 + 0.04}$$

$$x = \frac{A(s+0.5)((s+0.1)^2 + 0.04) + Bs((s+0.1)^2 + 0.04) + (Cs + D)s(s+0.5)}{s(s+0.5)((s+0.1)^2 + 0.04)}$$

$$x = \frac{As^3 + 0.7As^2 + 0.15As + 0.025A + Bs^3 + 0.2Bs^2 + 0.05Bs + Cs^3 + 0.5Cs^2 + Ds^2 + 0.5Ds}{s(s+0.5)((s+0.1)^2 + 0.04)}$$

$$A + B + C = 0$$

$$0.7A + 0.2B + 0.5C + D = 0$$

$$0.15A + 0.05B + 0.5D = 0$$

$$0.025A = 0.2$$

Solve the above 4 equations,

$$A = 8, B = -2, C = -6, D = -2.2$$

$$x = \frac{8}{s} - \frac{2}{s+0.5} - \frac{6s+2.2}{(s+0.1)^2+0.04}$$

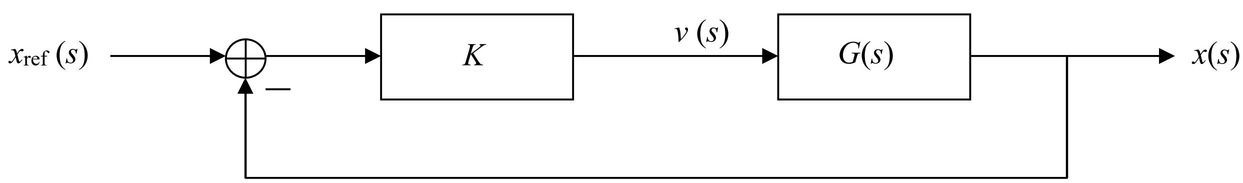
$$x = \frac{8}{s} - \frac{2}{s+0.5} - \frac{6(s+0.1)+1.6\left(\frac{0.2}{0.2}\right)}{(s+0.1)^2+0.04} = \frac{8}{s} - \frac{2}{s+0.5} - \frac{6(s+0.1)+8(0.2)}{(s+0.1)^2+0.04}$$

Take the inverse Laplace transformation,

$$x(t) = 8 - 2e^{-0.5t} - e^{-0.1t}(6\cos(0.2t) + 8\sin(0.2t))$$

$$x(t) = 8 - 2e^{-0.5t} - 10e^{-0.1t}\sin(0.2t + 0.6435)$$

(b) In order to control position of the actuator to the desired position, a proportional controller, *K*, is applied as shown in the below figure, determine the closed loop transfer function, the steady-state response when the reference is a unit step function. Roughly plot the root locus diagram, Nyquist diagram and bode diagram of the loop transfer function. For root locus diagram, determine zeros, poles, e, ∠, σ, break-in/away, cross over. For Nyquist diagram, determine the diagram at point O, and A, interception points with real/imaginary axis and the gains. For Bode diagram, draw bode diagram of each component of the loop transfer function. Determine the shortest dominant time constant from the proportional controller. (40 Points)



Solution

Closed loop transfer function,

$$G_c(s) = \frac{KG}{1+KG}$$

$$G_c(s) = \frac{4K}{20s^2 + 10s + 4K}$$

Steady-state response is determined from final value theorem,

$$x_{ss} = \lim_{s\to 0} s \times \frac{4K}{20s^2+10s+4K} \times \frac{1}{s} = 1$$

Root locus diagram

$$G = \frac{4}{20s^2+10s} = \frac{0.2}{s^2+0.5s} = \frac{0.2}{s(s+0.5)}$$

Zero = Φ

Poles = 0, -0.5

e = 2

$$\angle = \frac{180n}{2} = 90°, -90°$$

$$\sigma = \frac{-0.5}{2} = -0.25$$

Break-away point,

$$P(s) = s^2 + 0.5s + 0.2K$$

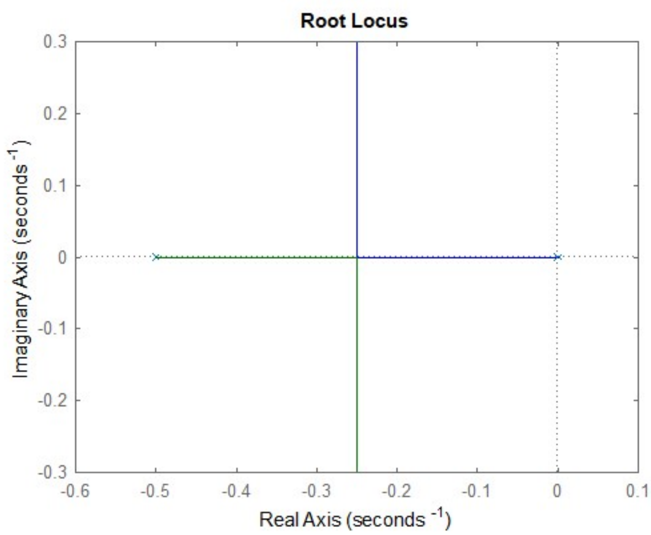
$$\acute{P} = 2s + 0.5$$

$$s = -0.25$$

Cross-over point,

$$P(\omega j) = -\omega^2 + 0.5\omega j + 0.2K = 0$$

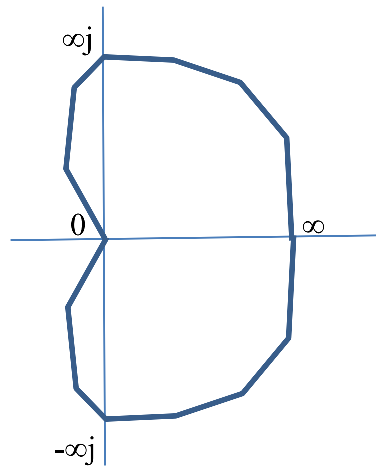
$$\omega = 0, K = 0$$



Nyquist Diagram

$$G = \frac{4}{20s^2 + 10s} = \frac{0.2}{s^2 + 0.5s} = \frac{0.2}{s(s + 0.5)}$$

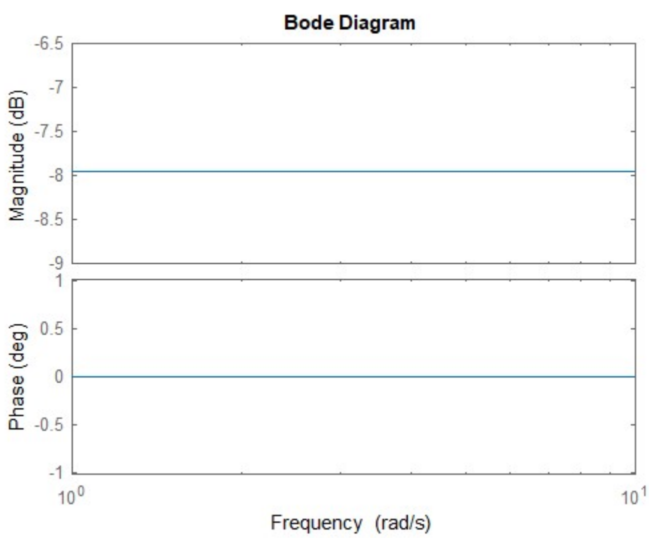
Point	G
0	$G(0,0j) \approx \lim_{s \to 0} \frac{4}{10s} = \infty, -\infty j$
A	$G(\omega j) = \frac{4}{-20\omega^2 + 10\omega j} = \frac{4(-20\omega^2 - 10\omega j)}{400\omega^4 + 100\omega^2}$
	$G(\infty j) = -0 - 0j$



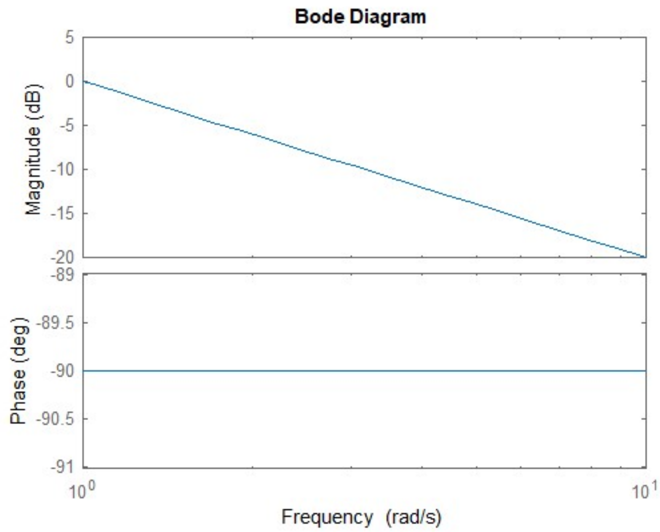
Bode Diagram

$$G = \frac{4}{20s^2 + 10s} = \frac{0.4}{s(2s+1)} = \frac{G_1}{G_2 G_3}$$

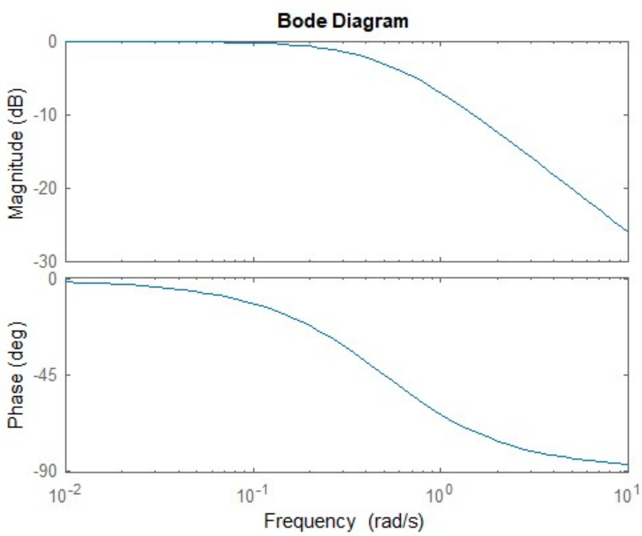
$G_1 = 0.4$



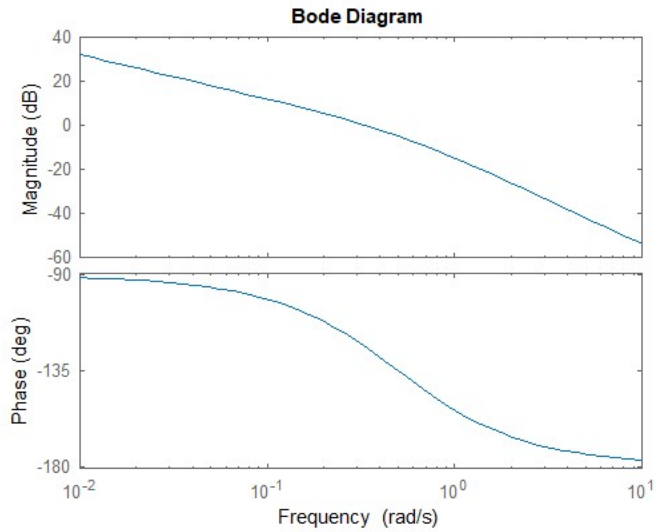
$$\frac{1}{G_2} = \frac{1}{s}$$



$$\frac{1}{G_3} = \frac{1}{2s+1}$$



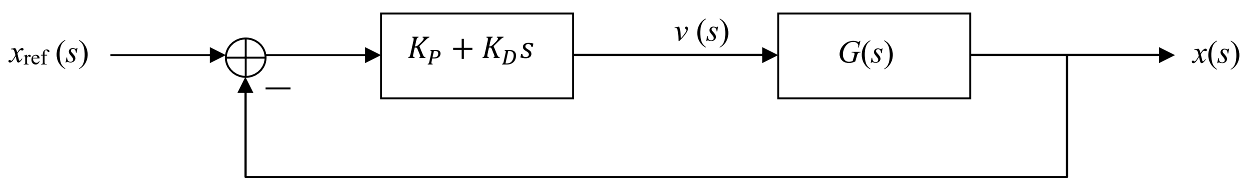
$$G = \frac{0.4}{s(2s+1)} = \frac{G_1}{G_2 G_3}$$



The shortest dominant time constant is obtained when the real time is at -0.25.

$$\tau = \frac{1}{0.25} = 4\ seconds$$

(c) In order to make the dominant time constant shorter, a proportional-derivative controller, $K_P + K_D s$, is applied as shown in the below figure, determine the closed loop transfer function, the steady-state response when the reference is a unit step function, and the proportional gain and derivative gain to achieve the time constant of 1 second with $1/\sqrt{2}$ damping ratio. (10 Points)



Solution

Closed loop transfer function,

$$G_c(s) = \frac{(K_P + K_D s)G}{1 + (K_P + K_D s)G}$$

$$G_c(s) = \frac{4K_D s + 4K_P}{20s^2 + (10 + 4K_D)s + 4K_P}$$

Steady-state response is determined from final value theorem,

$$x_{ss} = \lim_{s\to 0} s \times \frac{4K_D s + 4K_P}{20s^2 + (10 + 4K_D)s + 4K_P} \times \frac{1}{s} = 1$$

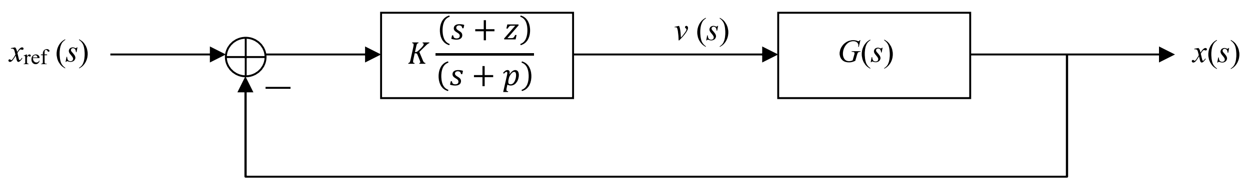
The desired characteristic equation is expressed by

$$(s + 1 + j)(s + 1 - j) = s^2 + 2s + 2 = 0$$

Equate characteristic equation with the desired characteristic equation,

$$20s^2 + (10 + 4K_D)s + 4K_P = 20(s^2 + 2s + 2) = 0$$
$$10 + 4K_D = 40$$
$$4K_p = 40$$
$$K_D = 7.5$$
$$K_P = 10$$

(d) If a lead compensator, $K\frac{(s+z)}{(s+p)}$, replaces the proportional-derivative controller as shown in the below figure, determine the closed loop transfer function, the steady-state response when the reference is a unit step function, and the lead compensator parameters, K, z, p to place 2 poles to have the time constant of 1 second with $1/\sqrt{2}$ damping ratio and the last pole at $-\sqrt{2}$. (10 Points)



Solution

Closed loop transfer function,

$$G_c(s) = \frac{K\frac{(s+z)}{(s+p)}G}{1 + K\frac{(s+z)}{(s+p)}G}$$

$$G_c(s) = \frac{4Ks + 4Kz}{20s^3 + (10 + 20p)s^2 + (10p + 4K)s + 4Kz}$$

Steady-state response is determined from final value theorem,

$$x_{ss} = \lim_{s\to 0} s \times \frac{4Ks + 4Kz}{20s^3 + (10 + 20p)s^2 + (10p + 4K)s + 4Kz} \times \frac{1}{s} = 1$$

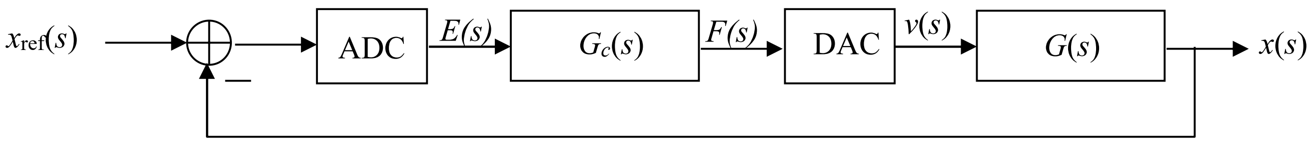
The desired characteristic equation is expressed by

$$(s + 1 + j)(s + 1 - j)\left(s + \sqrt{2}\right) = s^3 + \left(2 + \sqrt{2}\right)s^2 + \left(2 + 2\sqrt{2}\right)s + 2\sqrt{2} = 0$$

Equate characteristic equation with the desired characteristic equation,

$$20s^3 + (10 + 20p)s^2 + (10p + 4K)s + 4Kz = 20\left(s^3 + \left(2 + \sqrt{2}\right)s^2 + \left(2 + 2\sqrt{2}\right)s + 2\sqrt{2}\right) = 0$$
$$10 + 20p = 40 + 20\sqrt{2}$$
$$10p + 4K = 40 + 40\sqrt{2}$$
$$4Kz = 40\sqrt{2}$$
$$p = 2.914$$
$$K = 16.857$$
$$z = 0.839$$

(e) If a digital controller, $G_c(s)$, as shown in the block diagram below is used to control position of the actuator, design the controller by direct design method when the desired closed loop transfer function, T_c, is represented in s domain by $\frac{1-e^{-Ts}}{s}\cdot\frac{1}{s+1}$. The sampling time, T, is 0.1 second. Then determine the control signal at step k, $f(k)$, as a function of control signal and error, e, at the current and previous steps. (20 Points)



Solution

Determine the required closed loop transfer function with zero-order hold circuit,

$$T_{c2}(s)=\frac{T_c}{s}=\frac{1}{s}\cdot\frac{1}{s+1}=\frac{1}{s}-\frac{1}{s+1}$$

$$T_{c2}(z)=\frac{z}{z-1}-\frac{z}{z-e^{-0.1}}$$

$$T_c(z)=\frac{z-1}{z}T_{c2}(z)=1-\frac{z-1}{z-e^{-0.1}}=\frac{1-e^{-0.1}}{z-e^{-0.1}}$$

Determine the plant transfer function with zero-order hold circuit,

$$G_2(s)=\frac{1}{s}\cdot\frac{0.2}{s(s+0.5)}=-\frac{0.8}{s}+\frac{0.4}{s^2}+\frac{0.8}{s+0.5}$$

$$G_2(z)=-\frac{0.8z}{z-1}+\frac{0.04z}{(z-1)^2}+\frac{0.8z}{z-e^{-0.05}}$$

$$G(z)=\frac{z-1}{z}G_2(z)=-0.8+\frac{0.04}{z-1}+\frac{0.8(z-1)}{z-e^{-0.05}}=\frac{(-0.76+0.8e^{-0.05})z+0.8-0.84e^{-0.05}}{z^2-(1+e^{-0.05})z+e^{-0.05}}$$

$$G_c=\frac{T_c}{G(1-T_c)}=\frac{\frac{1-e^{-0.1}}{z-e^{-0.1}}}{\left(\frac{(-0.76+0.8e^{-0.05})z+0.8-0.84e^{-0.05}}{z^2-(1+e^{-0.05})z+e^{-0.05}}\right)\left(1-\frac{1-e^{-0.1}}{z-e^{-0.1}}\right)}$$

$$G_c=\frac{F}{E}=\frac{0.095z-0.091}{0.001z+0.001}$$

$$(0.001z+0.001)F=(0.095z-0.091)E$$

$$\left(0.001+\frac{0.001}{z}\right)F=\left(0.095-\frac{0.091}{z}\right)E$$

$$0.001f(k)+0.001f(k-1)=0.095e(k)-0.091e(k-1)$$

$$f(k)=f(k-1)+95e(k)-91e(k-1)$$