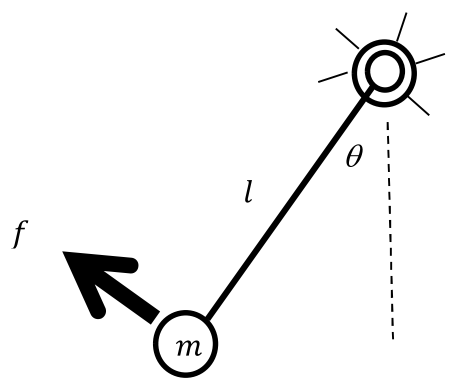
Final Examination **Control Theory AT74.10** **November 22, 2022**

Time: 13:00-15:00 hrs. Open Book
Marks: 100

Attempt all questions.

Consider the system in the midterm examination. A tangential force, f, is applied to a point mass, m, attached on a weightless rigid arm of length, l, from the frictionless joint as shown in the figure below. The dynamics of the system is obtained by considering the moment around the joint, $fl - mglsin\theta = ml^2\ddot{\theta}$. If the rotating angle, θ, is assumed small, $sin\theta \approx \theta$. Assume m = 5 kg, l = 2 m, g = 10 m/s^2.



The transfer function, G, from the tangential force to the rotating angle of this system is expressed by

$$G = \frac{\Theta}{F} = \frac{1}{mls^2 + mg} = \frac{1}{10s^2 + 50} = \frac{0.1}{s^2 + 5}$$

(a) Determine a state-space representation of the system in the Jordan form with the input, u, of the tangential force,f, and the output, y, of the rotating angle, θ. (20)

Solution

Transfer function,

$$G = \frac{\Theta}{F} = \frac{0.1}{s^2+5} = \frac{0.01\sqrt{5}j}{s+\sqrt{5}j} + \frac{-0.01\sqrt{5}j}{s-\sqrt{5}j} = \frac{\lambda+\gamma j}{s+\sigma-\omega j} + \frac{\lambda-\gamma j}{s+\sigma+\omega j} \quad (1)$$

Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\sigma^2+\omega^2) & -2\sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u] = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u] \quad \underline{(2)}$$

$$[y] = [2(\lambda\sigma - \omega\gamma j) \quad 2\lambda] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [0.1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underline{(3)}$$

(b) If the output is controlled to a constant reference rotating angle, θ_r, remodel the state-space system in (a) by taking into consideration the reference. (20)

Solution

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d = Ae + Bu + Ex_0 \tag{1}$$

For constant reference rotating angle,

$$[\theta_r] = [0.1 \quad 0]\begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} \tag{2}$$

Note that $x_{2r} = 0$.

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{1r} \\ x_2 - x_{2r} \end{bmatrix} = \begin{bmatrix} x_1 - x_{1r} \\ x_2 - x_{2r} \end{bmatrix} \tag{3}$$

$$E = A - A_r = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix} \tag{4}$$

Thus,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[u] + \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}\begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} \tag{5}$$

$$[y_e] = [0.1 \quad 0]\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{6}$$

(c) Determine the formula for the input of the tangential force, f , that places the second-order poles to have the time constant of 5 second with 0.5 damping ratio and the there is no error of the rotating angle. (30)

Solution

The desired characteristic equation is expressed by

$$s^2 + 0.4s + 0.16 = 0 \tag{1}$$

$$|sI - A + BG| = \begin{vmatrix} s & -1 \\ 5 + g_1 & s + g_2 \end{vmatrix} = s^2 + g_2 s + 5 + g_1 \tag{2}$$

(1) = (2),

$$G = [-4.84 \quad 0.4] \tag{3}$$

The gain for the exogeneous input is determined by

$$G_0 = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1}E \tag{4}$$

$$A + BG = \begin{bmatrix} 0 & 1 \\ -0.16 & -0.4 \end{bmatrix} \tag{5}$$

$$(A + BG)^{-1} = \frac{1}{0.16}\begin{bmatrix} -0.4 & -1 \\ 0.16 & 0 \end{bmatrix} \tag{6}$$

$$C(A + BG)^{-1} = \frac{1}{0.16}[0.1 \quad 0]\begin{bmatrix} -0.4 & -1 \\ 0.16 & 0 \end{bmatrix} = \frac{1}{0.16}[-0.04 \quad -0.1] \tag{7}$$

$$C(A + BG)^{-1}B = \frac{1}{0.16}[-0.04 \quad -0.1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{0.1}{0.16} \tag{8}$$

$$[C(A - BG)^{-1}B]^{-1} = -\frac{0.16}{0.1} = -1.6 \tag{9}$$

$$G_0 = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1}E = [-1.6]\frac{1}{0.16}[-0.04 \quad -0.1]\begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix} = [-5 \quad 0.4] \tag{10}$$

Thus,

$$u = 4.84e_1 - 0.4e_2 + 5x_{1r} - 0.4x_{2r} \tag{11}$$

(d) From the result in (b), remodel the system in the metastate form when all the state references are given and known. Design the reduced order observer by placing the observer pole to have the time constant of 1 second. (30)

Solution

From the result in (b)

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u] + \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} \tag{1}$$

$$[y_e] = [0.1 \quad 0] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{2}$$

The metastate form of the system is expressed by

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{x}_{1r} \\ \dot{x}_{2r} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -5 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ x_{1r} \\ x_{2r} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} [u] \tag{3}$$

$$\begin{bmatrix} y_e \\ x_{1r} \\ x_{2r} \end{bmatrix} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ x_{1r} \\ x_{2r} \end{bmatrix} \tag{4}$$

To design the reduced order observer, re-order the state variables,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{x}_{1r} \\ \dot{x}_{2r} \\ \hline \dot{e}_2 \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -5 & -5 & 0 & 0 \end{array}\right] \begin{bmatrix} e_1 \\ x_{1r} \\ x_{2r} \\ \hline e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline 1 \end{bmatrix} [u] \tag{5}$$

$$\begin{bmatrix} y_e \\ x_{1r} \\ x_{2r} \end{bmatrix} = \left[\begin{array}{ccc|c} 0.1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} e_1 \\ x_{1r} \\ x_{2r} \\ \hline e_2 \end{bmatrix} \tag{6}$$

$$\hat{x}_1 = \begin{bmatrix} \hat{e}_1 \\ \hat{x}_{1r} \\ \hat{x}_{2r} \end{bmatrix} = x_1 = \begin{bmatrix} e_1 \\ x_{1r} \\ x_{2r} \end{bmatrix} = C_1^{-1} y = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_e \\ x_{1r} \\ x_{2r} \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_e \\ x_{1r} \\ x_{2r} \end{bmatrix} \tag{7}$$

$$\hat{x}_2 = [\hat{e}_2] = Ly + z = [l_1 \quad l_2 \quad l_3] \begin{bmatrix} y_e \\ x_{1r} \\ x_{2r} \end{bmatrix} + [z] \tag{8}$$

$$[\dot{z}] = F\hat{x}_2 + \bar{g}y + Hu = [F][\hat{e}_2] + [\bar{g}_1 \quad \bar{g}_2 \quad \bar{g}_3] \begin{bmatrix} y_e \\ x_{1r} \\ x_{2r} \end{bmatrix} + [H][u] \tag{9}$$

$$F = A_{22} - LC_1 A_{12} = [0] - [l_1 \quad l_2 \quad l_3] \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = [-0.1 l_1] \tag{10}$$

Characteristic equation of the reduced order observer,

$$|sI - F| = s + 0.1l_1 = 0 \tag{11}$$

The desired characteristic equation of the reduced order observer is expressed by

$$s + 1 = 0 \tag{12}$$

(11) = (12),

$$l_1 = 10 \tag{13}$$

There is no constraint for l_2 and l_3, select

$$[l_1 \quad l_2 \quad l_3] = [10 \quad 0 \quad 0] \tag{14}$$

$$F = [-1] \tag{15}$$

$$\bar{g} = (A_{21} - LC_1A_{11})C_1^{-1} = \left([-5 \quad -5 \quad 0] - [10 \quad 0 \quad 0] \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

$$\bar{g} = [\bar{g}_1 \quad \bar{g}_2 \quad \bar{g}_3] = [-50 \quad -5 \quad -1] \tag{17}$$

$$H = B_2 - LC_1B_1 = [1] - [10 \quad 0 \quad 0] \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = [1] \tag{18}$$