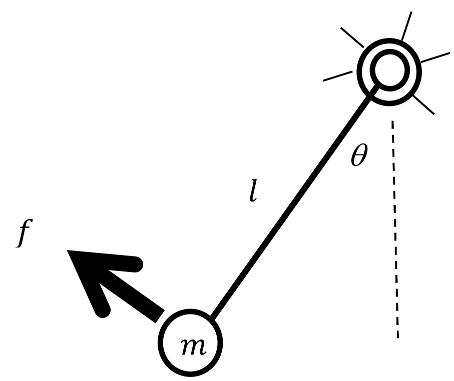
Midterm Examination **Control Theory AT74.10** **September 27, 2022**

Time: 9:00-12:00 hrs. Open Book
Marks: 100

Attempt all questions.

A tangential force, f, is applied to a point mass, m, attached on a weightless rigid arm of length, l, from the frictionless joint as shown in the figure below. The dynamics of the system is obtained by considering the moment around the joint, $fl - mglsin\theta = ml^2\ddot{\theta}$. If the rotating angle, θ, is assumed small, $sin\theta \approx \theta$. Assume m = 5 kg, l = 2 m, g = 10 m/s^2.



(a) If the tangential force is represented by $f = 2sin(\pi t/6)$ N and the initial condition of the rotating angle is set as $\theta(0) = -\pi/12$ rad and $\dot{\theta}(0) = \pi/18$ rad/s, determine the total response of the rotating angle in the form of the summation of two sinusoidal functions. (15 Points)

<u>Solution</u>

After linearization,

$$fl - mgl\theta = ml^2\ddot{\theta}$$

$$f = ml\ddot{\theta} + mg\theta$$

Take Laplace transformation,

$$F = ml\left(s^2\Theta - s\theta(0) - \dot{\theta}(0)\right) + mg\Theta$$

$$F + ml\left(s\theta(0) + \dot{\theta}(0)\right) = (mls^2 + mg)\Theta$$

$$\Theta = \frac{F + ml\left(s\theta(0) + \dot{\theta}(0)\right)}{mls^2 + mg}$$

Substitute the Laplace transformation of the force function and the parameters,

$$\Theta = \frac{(2\pi/6)}{(s^2 + (\pi/6)^2)(10s^2 + 50)} + \frac{10(-s\pi/12 + \pi/18)}{10s^2 + 50}$$

$$\Theta = \frac{As+B}{s^2+(\pi/6)^2} + \frac{Cs+D}{s^2+5} + \frac{-s\pi/12+\pi/18}{s^2+5}$$

$$(A+C)s^3 + (B+D)s^2 + (5A+(\pi/6)^2C)s + (5B+(\pi/6)^2D) = 2\pi/60$$

$$\Theta = \frac{0.022}{s^2+(\pi/6)^2} + \frac{-0.022}{s^2+5} + \frac{-s\pi/12+\pi/18}{s^2+5}$$

$$\Theta = \frac{0.042(\pi/6)}{s^2+(\pi/6)^2} + \frac{-0.262s+0.068\sqrt{5}}{s^2+5}$$

Take the inverse Laplace transformation,

$$\theta = 0.042sin(\pi t/6) - 0.262cos(\sqrt{5}t) + 0.068sin(\sqrt{5}t)$$

$$\theta = 0.042sin(\pi t/6) + 0.270sin(\sqrt{5}t - \text{atan}\,(3.853))$$

$$\theta = 0.042sin(\pi t/6) + 0.270sin(\sqrt{5}t - 1.317)$$

(b) Determine the transfer function, G, from the tangential force to the rotating angle of this system. (5 Points)

Solution

Neglect the initial condition,

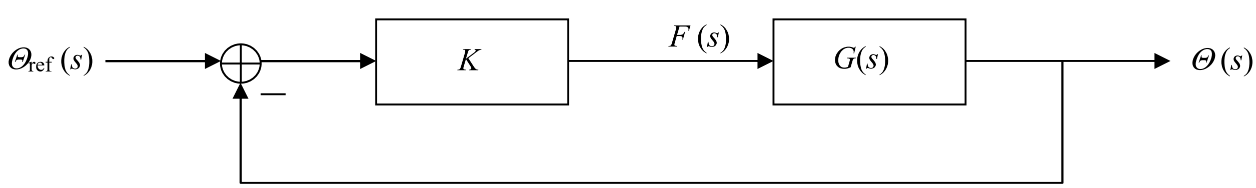
$$F = (mls^2 + mg)\Theta$$

$$G = \frac{\Theta}{F} = \frac{1}{mls^2+mg}$$

Substitute the parameters,

$$G = \frac{1}{10s^2+50}$$

(c) In order to control the rotating angle of the mass to the desired position, a proportional controller, K, is applied as shown in the below figure, determine the closed loop transfer function. Roughly plot the root locus diagram, Nyquist diagram and bode diagram of the loop transfer function. For root locus diagram, determine zeros, poles, e, ∠, σ, break-in/away, cross over. For Nyquist diagram, determine the diagram at point O, and A, interception points with real/imaginary axis and the gains. For Bode diagram, draw bode diagram of each component of the loop transfer function. Determine the shortest dominant time constant from the proportional controller. (20 Points)



Solution

Closed loop transfer function,

$$G_c(s) = \frac{KG}{1+KG}$$

$$G_c(s) = \frac{K}{10s^2+50+K}$$

Root locus diagram

$$G = \frac{1}{10s^2+50} = \frac{0.1}{s^2+5}$$

Zero = Φ

Poles = $\pm\sqrt{5}j$

e = 2

$$\angle = \frac{180n}{2} = 90°, -90°$$

$$\sigma = \frac{0}{2} = 0$$

Break-away point,

$$P(s) = 10s^2 + 50 + K$$

$$\acute{P} = 20s$$

$$s = 0$$

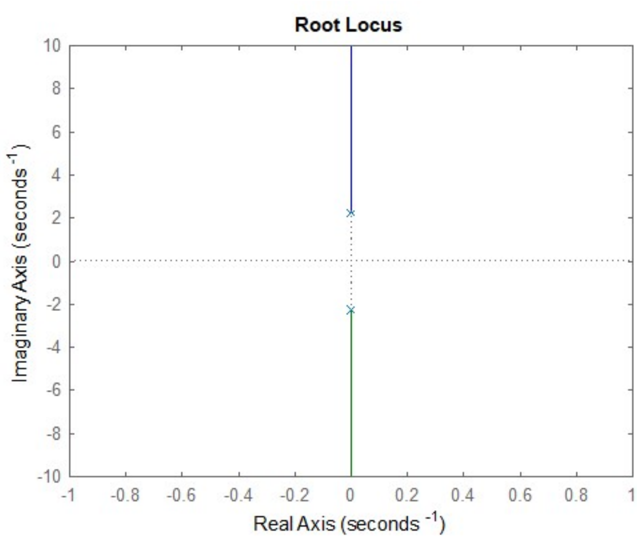
$$K = -50$$

There is no break-away point for positive gain.

Cross-over point,

$$P(\omega j) = -10\omega^2 + 50 + K = 0$$

There will always be ω for all positive gains.

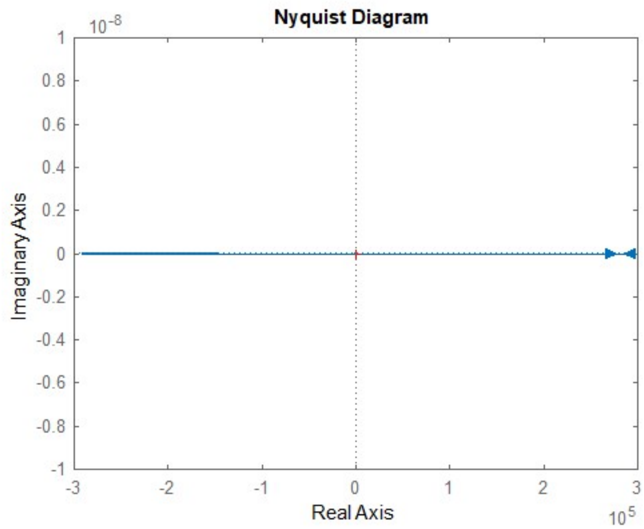


Nyquist Diagram

$$G = \frac{1}{10s^2 + 50} = \frac{0.1}{s^2 + 5}$$

Point	G
0	$G(0{,}0j) = 0.02$
A	$G(\omega j) = \frac{1}{-10\omega^2+5}$ The Nyquist diagram changes from ∞ to -∞ when $50 - 10\omega^2 = 0;\ \omega = \pm\sqrt{5}$ $G(\infty j) = -0$

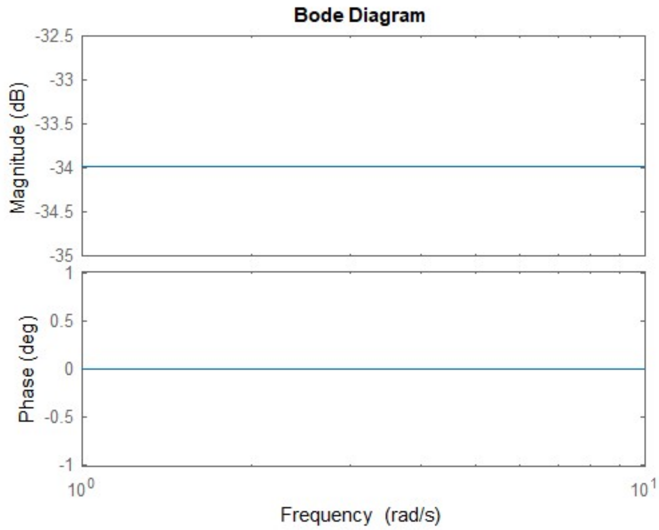
The Nyquist starts from 0.02 and then move to the right to ∞ then instantly jump to -∞ before approaching 0 along negative direction.



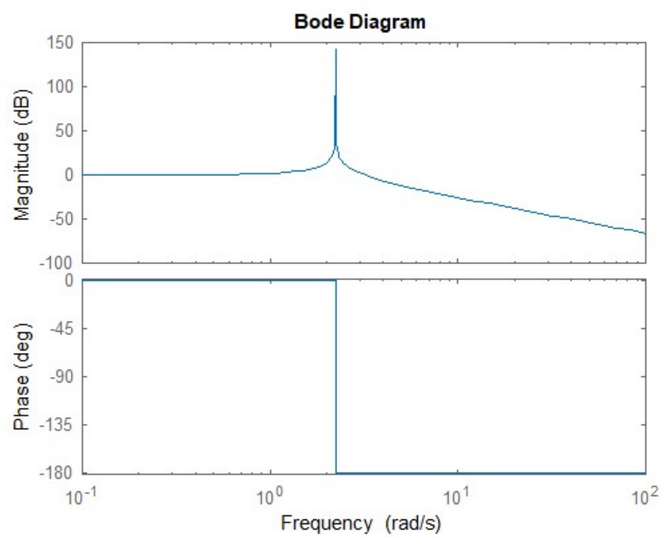
Bode Diagram

$$G = \frac{1}{10s^2 + 50} = \frac{0.02}{\left(\frac{s}{\sqrt{5}}\right)^2 + 1} = \frac{G_1}{G_2}$$

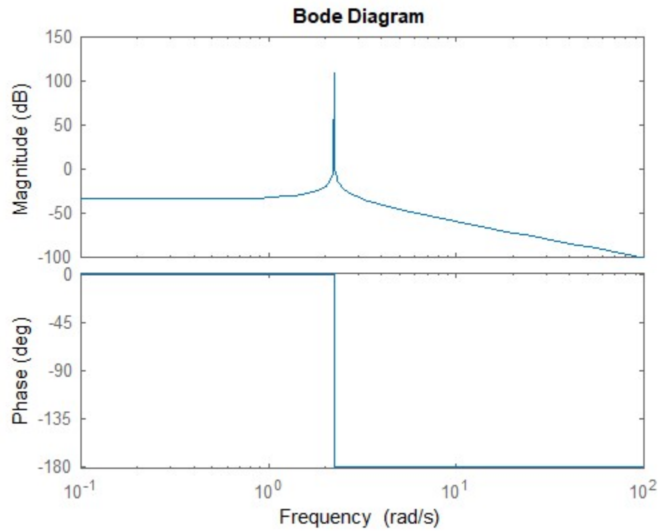
$G_1 = 0.02$



$$\frac{1}{G_2} = \frac{1}{\left(\frac{s}{\sqrt{5}}\right)^2 + 1}$$



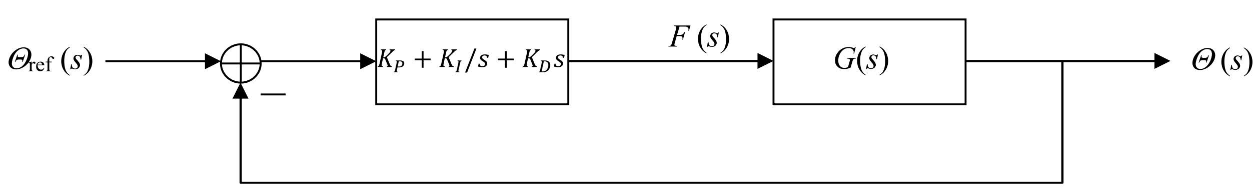
$$G = \frac{0.02}{\left(\frac{s}{\sqrt{5}}\right)^2 + 1} = \frac{G_1}{G_2}$$



Since the real term of the root of the characteristic equation is always 0, the dominant time constant is always infinity seconds by the proportional controller.

$$\tau = \frac{1}{0} = \infty \; seconds$$

(d) In order to make the system become asymptotically stable, a proportional-integral-derivative controller, $K_P + K_I/s + K_D s$, is applied as shown in the figure below, determine the closed loop transfer function, the steady-state response when the reference is a unit step function, and the proportional gain, integral gain and derivative gain that make the second-order poles have the time constant of 5 second with 0.5 damping ratio and the remaining pole locate at -10. (20 Points)



Solution

Closed loop transfer function,

$$G_c(s) = \frac{(K_P + K_I/s + K_D s)G}{1 + (K_P + K_I/s + K_D s)G}$$

$$G_c(s) = \frac{K_D s^2 + K_P s + K_I}{10s^3 + K_D s^2 + (50 + 4K_P)s + K_I}$$

Steady-state response is determined from final value theorem,

$$\theta_{ss} = \lim_{s \to 0} s \times \frac{K_D s^2 + K_P s + K_I}{10s^3 + K_D s^2 + (50 + 4K_P)s + K_I} \times \frac{1}{s} = 1$$

The desired characteristic equation is expressed by

$$(s^2 + 0.4s + 0.16)(s + 10) = s^3 + 10.4s^2 + 4.16s + 1.6 = 0$$

Equate characteristic equation with the desired characteristic equation,

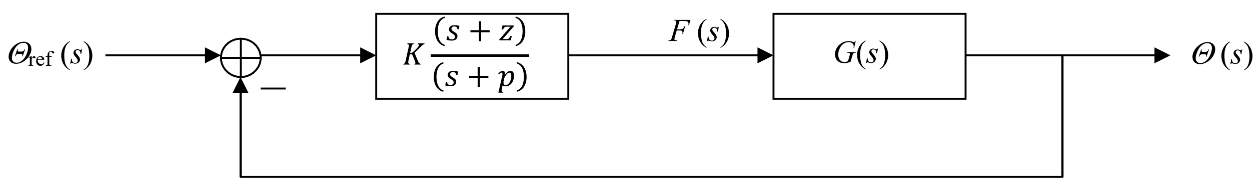
$$s^3 + 0.1K_D s^2 + (5 + 0.4K_P)s + 0.1K_I = s^3 + 10.4s^2 + 4.16s + 1.6 = 0$$

$$K_P = -2.1$$

$$K_I = 16$$

$$K_D = 104$$

(e) If a compensator, $K\frac{(s+z)}{(s+p)}$, is applied as shown in the below figure, determine the closed loop transfer function, and the compensator parameters, K, z, p that make the second-order poles have the time constant of 5 second with 0.5 damping ratio and the remaining pole locate at -10. Determine the steady-state response when the reference is a unit step function. (10 Points)



Solution

Closed loop transfer function,

$$G_c(s) = \frac{K\frac{(s+z)}{(s+p)}G}{1 + K\frac{(s+z)}{(s+p)}G}$$

$$G_c(s) = \frac{Ks + Kz}{10s^3 + 10ps^2 + (50 + K)s + 50p + Kz}$$

The desired characteristic equation is expressed by

$$(s^2 + 0.4s + 0.16)(s + 10) = s^3 + 10.4s^2 + 4.16s + 1.6 = 0$$

Equate characteristic equation with the desired characteristic equation,

$$s^3 + ps^2 + (5 + 0.1K)s + 5p + 0.1Kz = s^3 + 10.4s^2 + 4.16s + 1.6 = 0$$

$$p = 10.4$$

$$K = -8.4$$

$$z = 60$$

Steady-state response is determined from final value theorem,

$$\theta_{ss} = \lim_{s\to 0} s \times \frac{Ks + Kz}{10s^3 + 10ps^2 + (50 + K)s + 50p + Kz} \times \frac{1}{s} = \frac{Kz}{50p + Kz} = -31.5$$

This steady-state response indicates that the mass rotates 31.5 rad or about 5 revolutions before it comes to the steady-state to achieve the desired transient requirement. This result is not feasible since the dynamics model was linearized by assuming that $sin\theta \approx \theta$ which is valid only for small rotating angle.

(f) If the requirement for the compensator in (e) is changed to have the steady-state error of 0.1 when the reference is a unit step function and the compensator pole is placed at -0.1 while the compensator zero is placed at -0.5, determine the compensator parameters, K, and the characteristic equation of the compensated system. (10 Points)

Solution

Closed loop transfer function,

$$G_c(s) = \frac{K\dfrac{(s+z)}{(s+p)}G}{1+K\dfrac{(s+z)}{(s+p)}G}$$

$$G_c(s) = \frac{Ks+Kz}{10s^3+10ps^2+(50+K)s+50p+Kz}$$

Steady-state response is determined from final value theorem,

$$\theta_{ss} = \lim_{s\to 0} s \times \frac{Ks+Kz}{10s^3+10ps^2+(50+K)s+50p+Kz} \times \frac{1}{s} = \frac{Kz}{50p+Kz} = \frac{0.5K}{5+0.5K} = 0.9$$

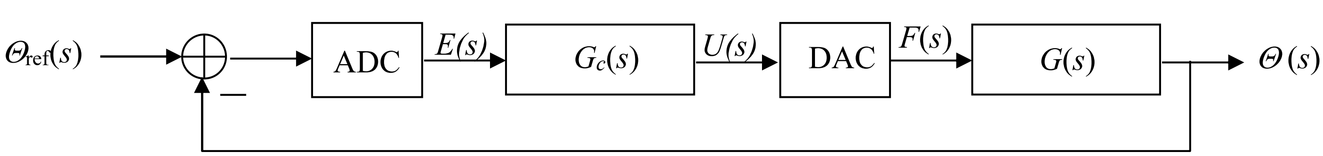
$$K = 90$$

Characteristic equation,

$$10s^3+10ps^2+(50+K)s+50p+Kz = 10s^3+s^2+140s+50 = 0$$

$$s^3+0.1s^2+14s+5 = 0$$

(g) If a digital controller, $G_c(s)$, as shown in the block diagram below is used to control the rotating angle, design the controller by direct design method when the desired closed loop transfer function, T_c, is represented in *s* domain by $\frac{1-e^{-Ts}}{s}\cdot\frac{1}{s+0.2}$. The sampling time, *T*, is 0.1 second. Then determine the control signal at step *k*, *u* (*k*), as a function of control signal and error, *e*, at the current and previous steps. (20 Points)



Solution

Determine the required closed loop transfer function with zero-order hold circuit,

$$T_{c2}(s) = \frac{T_c}{s} = \frac{1}{s} \cdot \frac{1}{s+0.2} = \frac{5}{s} - \frac{5}{s+0.2}$$

$$T_{c2}(z) = \frac{5z}{z-1} - \frac{5z}{z-e^{-0.02}}$$

$$T_c(z) = \frac{z-1}{z} T_{c2}(z) = 5 - \frac{5z-5}{z-e^{-0.02}} = \frac{5-5e^{-0.02}}{z-e^{-0.02}} = \frac{0.10}{z-0.98}$$

Determine the plant transfer function with zero-order hold circuit,

$$G_2(s) = \frac{1}{s} \cdot \frac{1}{10s^2+50} = \frac{1}{s} \cdot \frac{0.1}{s^2+5} = \frac{0.02}{s} - \frac{0.02}{s^2+5}$$

$$G_2(z) = \frac{0.02z}{z-1} + \frac{0.02(z^2 - cos(0.22)z)}{z^2 - 2cos(0.22)z + 1} = \frac{0.02z}{z-1} + \frac{0.02z^2 - 0.22z}{z^2 - 1.95z + 1}$$

$$G(z) = \frac{z-1}{z} G_2(z) = 0.02 + \frac{0.02z^2 - 0.24z + 0.22}{z^2 - 1.95z + 1} = \frac{0.04z^2 - 0.28z + 0.24}{z^2 - 1.95z + 1}$$

$$G_c = \frac{T_c}{G(1-T_c)} = \frac{\frac{0.10}{z-0.98}}{\left(\frac{0.04z^2 - 0.28z + 0.24}{z^2 - 1.95z + 1}\right)\left(1 - \frac{0.10}{z-0.98}\right)}$$

$$G_c = \frac{U}{E} = \frac{0.1z^2 - 0.20z + 0.10}{0.04z^3 - 0.32z^2 + 0.54z - 0.26}$$

$$(0.04z^3 - 0.32z^2 + 0.54z - 0.26)U = (0.1z^2 - 0.20z + 0.10)E$$

$$\left(0.04 - \frac{0.32}{z} + \frac{0.54}{z^2} - \frac{0.26}{z^3}\right)U = \left(\frac{0.1}{z} - \frac{0.20}{z^2} + \frac{0.10}{z^3}\right)E$$

$$0.04u(k) - 0.32u(k-1) + 0.54u(k-2) - 0.26u(k-3)$$
$$= 0.1e(k-1) - 0.20e(k-2) + 0.10e(k-3)$$

$$u(k) = 8u(k-1) - 13.5u(k-2) + 6.5u(k-3) + 2.5e(k-1) - 5e(k-2) + 2.5e(k-3)$$