Final Examination **Control Theory AT74.10** **November 27, 2023**

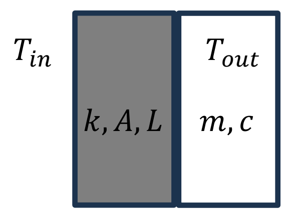
Time: 9:00-11:00 hrs. Open Book
Marks: 100

Attempt all questions.

Consider the system in the midterm examination. Temperature of a heater, T_{in}, is used to control temperature of a block, T_{out}. The relation between the heater temperature and the block temperature is determined from the heat transfer rate through medium material as expressed by

$$\frac{kA(T_{in} - T_{out})}{L} = mc\frac{dT_{out}}{dt}$$

When $k = 45\ W/(m°C)$ is medium material conductivity, $A = 0.01\ m^2$ is contacting surface area, $L = 0.03\ m$ is medium material thickness, $m = 1\ kg$ is block mass, $c = 450\ J/(kg°C)$ is block heat capacity, and t is time.



the transfer function of the system of this system is expressed by

$$G(s) = \frac{T_{out}}{T_{in}} = \frac{kA/L}{mcs + kA/L} = \frac{1}{30s + 1}$$

(a) Determine a state-space representation of the system when $y = x_1 = T_{out}$ and $u = T_{in}$. (10)

Solution

$$(30s + 1)T_{out} = T_{in} \quad (1)$$

Thus,

$$[\dot{x}_1] = \left[-\frac{1}{30}\right][x_1] + \left[\frac{1}{30}\right][u] \quad \underline{(2)}$$

$$[y] = [1][x_1] \quad \underline{(3)}$$

(b) If T_{out} is controlled to a constant reference temperature, T_{out_ref}, and it is disturbed by an external disturbance, T_d, as expressed by the transfer function $\frac{T_{out}}{T_d} = \frac{1}{s}$, remodel the state-space system in (a) by taking into consideration the reference and the external disturbance. (10)

Solution

The relation between T_d and T_{out} follows

$$\dot{T}_{out} = T_d \tag{1}$$

With exogenous input, the system is remodeled as,

$$\dot{e} = Ae + Bu + (A - A_r)x_r + Fx_d = \left[-\frac{1}{30}\right][e] + \left[\frac{1}{30}\right][u] + \left[-\frac{1}{30}\right]\left[T_{out\_ref}\right] + [T_d] \tag{2}$$

When $e = T_{out} - T_{out\_ref}$.

$$[y_e] = [1][e] \tag{3}$$

(c) Determine the formula for the input of the heater temperature, T_{in} , that places the pole to have the time constant of 5 seconds and there is no error of the block temperature. (30)

Solution

The desired characteristic equation is expressed by

$$s + 0.2 = 0 \tag{1}$$

$$|sI - A + BG| = \left|s + \frac{1}{30} + \frac{g}{30}\right| = s + 0.2 \tag{2}$$

$$G = [5] \tag{3}$$

The gain for the exogeneous input is determined by

$$G_0 = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1}E \tag{4}$$

$$A - BG = [-0.2] \tag{5}$$

$$(A - BG)^{-1} = [-5] \tag{6}$$

$$C(A - BG)^{-1} = [1][-5] = [-5] \tag{7}$$

$$C(A - BG)^{-1}B = [-5]\left[\frac{1}{30}\right] = \left[-\frac{1}{6}\right] \tag{8}$$

$$[C(A - BG)^{-1}B]^{-1} = [-6] \tag{9}$$

$$G_0 = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1}E = [-6][-5]\left[-\frac{1}{30} \quad 1\right] = [-1 \quad 30] \tag{10}$$

Thus,

$$u = T_{in} = -5e + T_{out\_ref} - 30T_d \tag{11}$$

(d) Determine the formula for the input of the heater temperature, T_{in} , that minimizes the cost function expressed by

$$V = \int_0^\infty \left(4\left(T_{out} - T_{out\_ref}\right)^2 + v^2\right) dt$$

when $u = v + \bar{u}$, and $\bar{u}$ is used to keep the temperature error $\left(T_{out} - T_{out\_ref}\right)$, at zero at steady state under the reference, T_{out_ref} and the disturbance, T_d. Then determine the characteristic equation of the regulated system. (30)

Solution

$$Q = [4] \tag{1}$$

$$R = [1] \tag{2}$$

Determine the gain for the state error

$$G = R^{-1}B^t\bar{M}_1 \tag{3}$$

$$R^{-1}B^t\bar{M}_1 = [1]^{-1}\left[\frac{1}{30}\right]^t[m_1] \tag{4}$$

$$R^{-1}B^t\bar{M}_1 = \left[\frac{m_1}{30}\right] \tag{5}$$

When

$$0 = -\dot{\bar{M}}_1 = \bar{M}_1A + A^t\bar{M}_1 - \bar{M}_1BR^{-1}B^t\bar{M}_1 + Q \tag{6}$$

Substitute all the concerned matrices,

$$[0] = [m_1]\left[-\frac{1}{30}\right] + \left[-\frac{1}{30}\right][m_1] - [m_1]\left[\frac{1}{30}\right]\left[\frac{m_1}{30}\right] + [4] \tag{7}$$

$$[0] = \left[4 - \frac{2m_1}{30} - \frac{m_1^2}{900}\right] \tag{8}$$

Thus,

$$[m_1] = [37.08] \tag{9}$$

$$G = R^{-1}B^t\bar{M}_1 = [1.24] \tag{10}$$

$$A_c = A - BG = \left[-\frac{1}{30}\right] - \left[\frac{1}{30}\right][1.24] = \left[-\frac{2.24}{30}\right] = [-0.07] \tag{11}$$

Determine the gain of the exogeneous input,

$$G_0 = R^{-1}B^t\bar{M}_2 \tag{12}$$

$$R^{-1}B^t\bar{M}_2 = [1]^{-1}\left[\frac{1}{30}\right]^t[m_{21} \quad m_{22}] \tag{13}$$

$$R^{-1}B^t\bar{M}_2 = \left[\frac{m_{21}}{30} \quad \frac{m_{22}}{30}\right] \tag{14}$$

When

$$0 = -\dot{\bar{M}}_2 = \bar{M}_1E + \bar{M}_2A_0 + (A^t - \bar{M}_1BR^{-1}B^t)\bar{M}_2 \tag{15}$$

Substitute all the concerned matrices,

$$[0 \quad 0] = [37.08]\left[-\frac{1}{30} \quad 1\right] + [m_{21} \quad m_{22}]\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + [-0.07][m_{21} \quad m_{22}] \tag{16}$$

$$[0 \quad 0] = \left[-\frac{37.08}{30} \quad 37.08\right] + [-0.07m_{21} \quad -0.07m_{22}] \tag{17}$$

$$[m_{21} \quad m_{22}] = [-17.66 \quad 529.71] \tag{18}$$

$$G_0 = R^{-1}B^t\bar{M}_2 = [-0.59 \quad 17.66] \tag{19}$$

Thus,

$$u = -[1.24][e] - [-0.59 \quad 17.66]\begin{bmatrix} T_{out\_ref} \\ T_d \end{bmatrix} \tag{20}$$

Characteristic equation of the regulated system is determined from

$$|sI - A_C| = |s + 0.07| \tag{21}$$

(e) From the result in (b), remodel the system in the metastate form. Design the reduce order observer by placing the observer pole to have the time constant of 1 second when both T_{out} and T_{out_ref} are obtained by measurement. (20)

Solution

From the result in (b)

$$\dot{e} = \left[-\frac{1}{30}\right][e] + \left[\frac{1}{30}\right][u] + \left[-\frac{1}{30}\right][T_{out\_ref}] + [T_d] \tag{1}$$

$$[y_e] = [1][e] \tag{2}$$

$$[y_{Tout\_ref}] = [1][T_{out\_ref}] \tag{3}$$

The metastate form of the system is expressed by

$$\begin{bmatrix} \dot{e} \\ \dot{T}_{out\_ref} \\ \dot{T}_d \end{bmatrix} = \begin{bmatrix} -\frac{1}{30} & -\frac{1}{30} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} e \\ T_{out\_ref} \\ T_d \end{bmatrix} + \begin{bmatrix} \frac{1}{30} \\ 0 \\ 0 \end{bmatrix}[u] \tag{4}$$

$$\begin{bmatrix} y_e \\ y_{Tout\_ref} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} e \\ T_{out\_ref} \\ T_d \end{bmatrix} \tag{5}$$

$$\hat{x}_1 = \begin{bmatrix} \hat{e} \\ \hat{T}_{out\_ref} \end{bmatrix} = x_1 = \begin{bmatrix} e \\ T_{out\_ref} \end{bmatrix} = C_1^{-1}y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} y_e \\ y_{Tout\_ref} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_e \\ y_{Tout\_ref} \end{bmatrix} \tag{6}$$

$$\hat{x}_2 = [\hat{T}_d] = Ly + z = [l_1 \quad l_2]\begin{bmatrix} y_e \\ y_{Tout\_ref} \end{bmatrix} + [z] \tag{7}$$

$$[\dot{z}] = F\hat{x}_2 + \bar{g}y + Hu = [F][\hat{T}_d] + [\bar{g}_1 \quad \bar{g}_2]\begin{bmatrix} y_e \\ y_{Tout\_ref} \end{bmatrix} + [H][u] \tag{8}$$

$$F = A_{22} - LC_1A_{12} = [0] - [l_1 \quad l_2]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [-l_1] \tag{9}$$

Characteristic equation of the reduced order observer,

$$|sI - F| = s + l_1 = 0 \tag{10}$$

The desired characteristic equation of the reduced order observer is expressed by

$$s + 1 = 0 \tag{11}$$

(11) = (12),

$$l_1 = 1 \tag{12}$$

There is no constraint for l_2, select

$$[l_1 \quad l_2] = [1 \quad 0] \tag{13}$$

$$F = [-1] \qquad (14)$$

$$\bar{g} = (A_{21} - LC_1A_{11})C_1^{-1} = \left([0 \quad 0] - [1 \quad 0]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-\frac{1}{30} & -\frac{1}{30}\\0 & 0\end{bmatrix}\right)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \qquad (15)$$

$$\bar{g} = [\bar{g}_1 \quad \bar{g}_2] = \begin{bmatrix}\frac{1}{30} & \frac{1}{30}\end{bmatrix} \qquad (16)$$

$$H = B_2 - LC_1B_1 = [0] - [1 \quad 0]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{30}\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{30}\end{bmatrix} \qquad (17)$$