Midterm Examination **Control Theory AT74.10** **September 26, 2023**

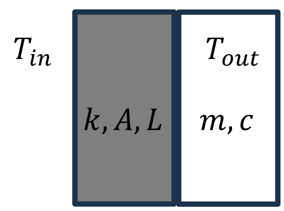
Time: 9:00-11:00 hrs. Open Book
Marks: 100

Attempt all questions.

Temperature of a heater, T_{in}, is used to control temperature of a block, T_{out}. The relation between the heater temperature and the block temperature is determined from the heat transfer rate through medium material as expressed by

$$\frac{kA(T_{in} - T_{out})}{L} = mc\frac{dT_{out}}{dt}$$

When $k = 45\,W/(m°C)$ is medium material conductivity, $A = 0.01\,m^2$ is contacting surface area, $L = 0.03\,m$ is medium material thickness, $m = 1\,kg$ is block mass, $c = 450\,J/(kg°C)$ is block heat capacity, and t is time.



(a) Determine the transfer function of the system, $G(s) = \frac{T_{out}(s)}{T_{in}(s)}$, what are the magnitude ratio of T_{in} and T_{out} at the steady state and the system time constant. (10 Points)

Solution

Take Laplace transformation and rearrange the relation,

$$\left(mcs + \frac{kA}{L}\right)T_{out} = \frac{kA}{L}T_{in}$$

$$G(s) = \frac{T_{out}}{T_{in}} = \frac{kA/L}{mcs + kA/L} = \frac{(45 \times 0.01/0.03)}{(1 \times 450s + 45 \times 0.01/0.03)} = \frac{15}{450s + 15}$$

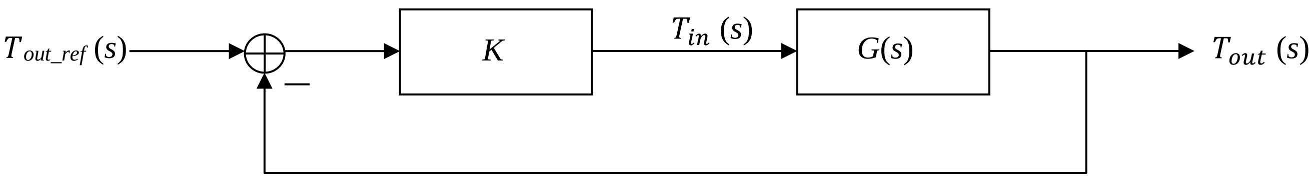
$$G(s) = \frac{T_{out}}{T_{in}} = \frac{15}{450s + 15} = \frac{1}{30s + 1} = \frac{1/30}{s + 1/30}$$

Thus, the magnitude ratio at the steady state becomes

$$G_{ss} = \frac{1}{0 + 1} = 1$$

The time constant is 50 seconds.

(b) In order to shorten the system time constant, a proportional controller, K, is applied as shown in the below figure, determine the closed loop transfer function. Roughly plot the root locus diagram, Nyquist diagram and bode diagram of the loop transfer function. For root locus diagram, determine zeros, poles, e, ∠, σ, break-in/away, cross over. For Nyquist diagram, determine the diagram at point O, and A, interception points with real/imaginary axis and the gain. For Bode diagram, draw bode diagram of each component of the loop transfer function. Determine the proportional gain that shorten the time constant to 5 seconds and the magnitude ratio of T_{out_ref} and T_{out} at the steady state. (20 Points)



<u>Solution</u>

Closed loop transfer function,

$$G_c(s) = \frac{KG}{1 + KG}$$

$$G_c(s) = \frac{K}{30s + 1 + K}$$

Root locus diagram

$$G = \frac{1}{30s + 1} = \frac{1/30}{s + 1/30}$$

Zero = Φ

Poles = -0.03

e = 1

$$\angle = \frac{180n}{1} = 180°$$

$$\sigma = \frac{-1/30}{1} = -1/30$$

Break-away point,

$$P(s) = 30s + 1 + K$$

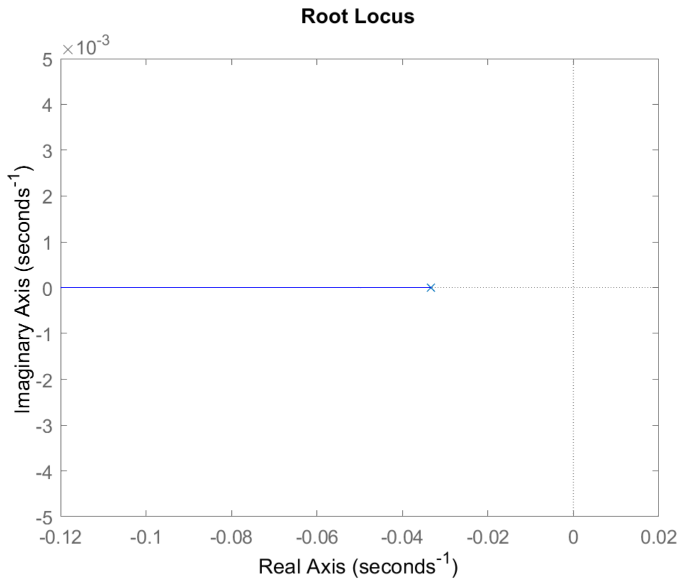
$$\acute{P} = 50$$

There is no break-away/in point.

Cross-over point,

$$P(\omega j) = 30\omega j + 1 + K = 0$$

There is no cross-over point for all positive gains.

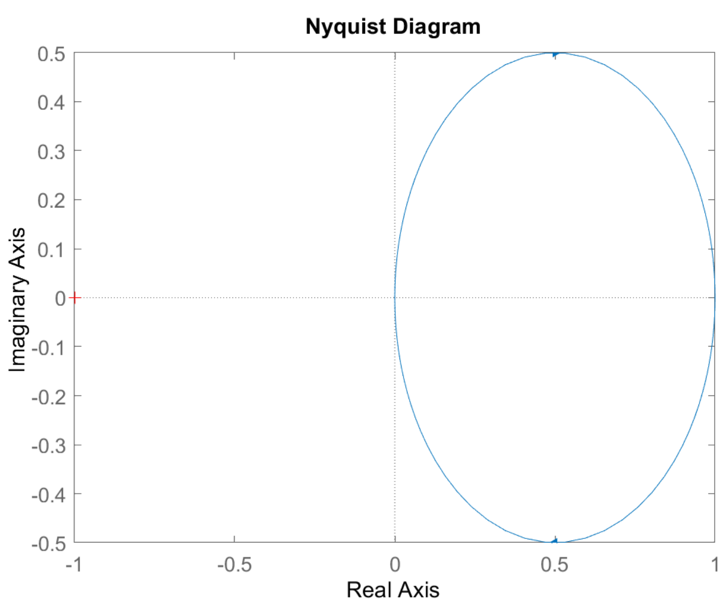


Nyquist Diagram

$$G = \frac{1}{30s+1} = \frac{1/30}{s+1/30}$$

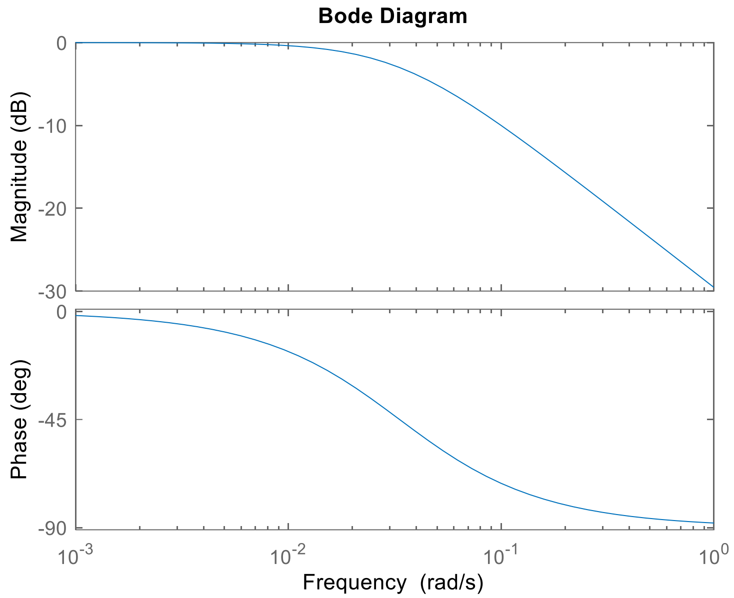
Point	G
0	$G(0,0j) = 1$
A	$G(\omega j) = \frac{1-30\omega j}{1+900\omega^2}$
	$G(\infty j) = 0 - 0j$

The Nyquist starts from 1 and then move to the right before approaching 0 along positive real and negative imaginary direction.



Bode Diagram

$$G = \frac{1}{30s + 1}$$



Characteristic equation of the desired time constant of 5 seconds $5s + 1 = 0$ or $s + 0.2 = 0$.

Characteristic equation of the system with proportional controller $30s + 1 + K = 0$ or $s + 1/30 + K/30 = 0$

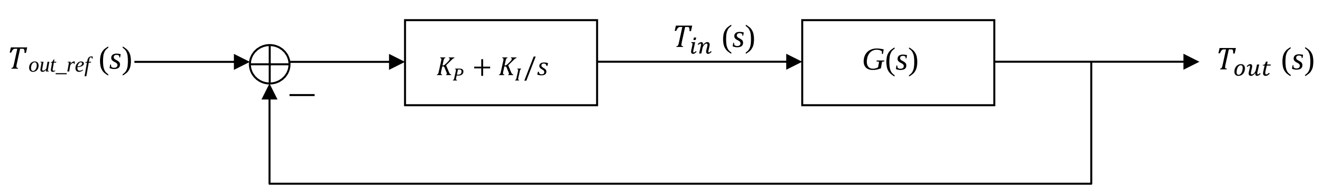
$$s + 1/30 + \frac{K}{30} = s + 0.2$$

$$K = 5$$

Thus, the magnitude ratio at the steady state becomes

$$G_{ss} = \frac{K}{0 + 1 + K} = \frac{5}{1 + 5} = 0.83$$

(c) If the desired magnitude ratio of T_{out_ref} and T_{out} at the steady state is 1 with time constant of 5 seconds and critical damp, a proportional-integral controller, $K_P + K_I/s$, is applied as shown in the figure below, determine the closed loop transfer function, the proportional gain, and the integral gain. (20 Points)



<u>Solution</u>

Closed loop transfer function,

$$G_c(s) = \frac{(K_P + K_I/s)G}{1 + (K_P + K_I/s)G}$$

$$G_c(s) = \frac{K_P s + K_I}{30s^2 + (K_P + 1)s + K_I} = \frac{K_P s/30 + K_I/30}{s^2 + (K_P + 1)s/30 + K_I/30}$$

The magnitude ratio at the steady state becomes

$$G_{ss} = \frac{0 + K_I}{0 + 0 + K_I} = 1$$

The desired characteristic equation is expressed by

$$(s + 0.2)^2 = s^2 + 0.4s + 0.04 = 0$$

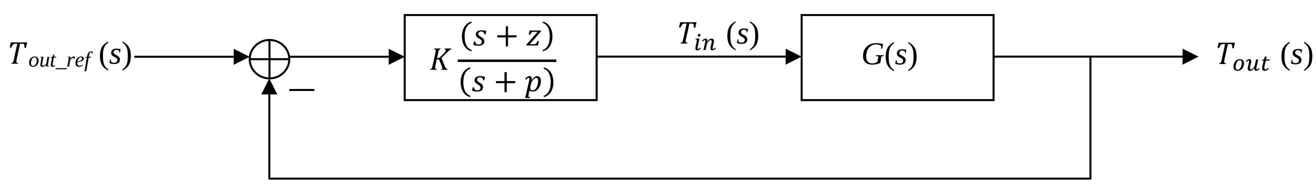
Equate characteristic equation with the desired characteristic equation,

$$s^2 + (K_P + 1)s/30 + K_I/30 = s^2 + 0.4s + 0.04 = 0$$

$$K_P = 11$$

$$K_I = 1.2$$

(d) If a compensator, $K\frac{(s+z)}{(s+p)}$, is applied as shown in the below figure, determine the closed loop transfer function, and the compensator parameters, K, z, p that make critical damp with time constant of 5 seconds and the magnitude ratio of T_{out_ref} and T_{out} at the steady state of 0.99. (20 Points)



Solution

Closed loop transfer function,

$$G_c(s) = \frac{K\frac{(s+z)}{(s+p)}G}{1 + K\frac{(s+z)}{(s+p)}G}$$

$$G_c(s) = \frac{Ks + Kz}{30s^2 + (30p + 1 + K)s + p + Kz} = \frac{Ks/30 + Kz/30}{s^2 + (30p + 1 + K)s/30 + (p + Kz)/30}$$

The desired characteristic equation is expressed by

$$(s + 0.2)^2 = s^2 + 0.4s + 0.04 = 0$$

The magnitude ratio at the steady state becomes

$$G_{ss} = \frac{0 + Kz}{0 + 0 + p + Kz} = 0.99$$

Thus

$$p = \frac{Kz}{99}$$

$$s^2 + \frac{(30p + 1 + K)s}{30} + \frac{p + Kz}{30} = s^2 + \left(\frac{Kz}{99} + \frac{1}{30} + \frac{K}{30}\right)s + \frac{Kz}{29.7}$$

Equate characteristic equation with the desired characteristic equation,

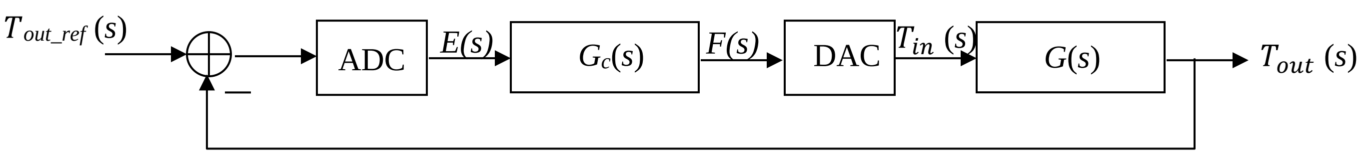
$$s^2 + \left(\frac{Kz}{99} + \frac{1}{30} + \frac{K}{30}\right)s + \frac{Kz}{29.7} = s^2 + 0.4s + 0.04 = 0$$

$$K = 10.64$$

$$z = 0.11$$

$$p = 0.012$$

(e) If a digital controller, $G_c(s)$, as shown in the block diagram below is used to control the block temperature, design the controller by direct design method when the desired closed loop transfer function, T_c, is represented in s domain by $\frac{1-e^{-Ts}}{s} \cdot \frac{0.2}{s+0.2}$. The sampling time, T, is 0.05 second. Then determine the control signal at step k, $f(k)$, as a function of control signal and error, e, at the current and previous steps. (30 Points)



Solution

Determine the required closed loop transfer function with zero-order hold circuit,

$$T_{c2}(s) = \frac{T_c}{s} = \frac{1}{s} \cdot \frac{0.2}{s + 0.2} = \frac{1}{s} - \frac{1}{s + 0.2}$$

$$T_{c2}(z) = \frac{z}{z - 1} - \frac{z}{z - e^{-0.01}}$$

$$T_c(z) = \frac{z - 1}{z} T_{c2}(z) = 1 - \frac{z - 1}{z - e^{-0.01}} = \frac{1 - e^{-0.01}}{z - e^{-0.01}} = \frac{0.01}{z - 0.9900}$$

Determine the plant transfer function with zero-order hold circuit,

$$G_2(s) = \frac{1}{s} \cdot \frac{0.03}{s + 0.03} = \frac{1}{s} - \frac{1}{s + 0.03}$$

$$G_2(z) = \frac{z}{z - 1} - \frac{z}{z - e^{-0.0015}}$$

$$G(z) = \frac{z - 1}{z} G_2(z) = 1 - \frac{z - 1}{z - e^{-0.0015}} = \frac{1 - e^{-0.0015}}{z - e^{-0.0015}} = \frac{0.0015}{z - 0.9985}$$

$$G_c = \frac{T_c}{G(1 - T_c)} = \frac{\frac{0.01}{z - 0.9900}}{\left(\frac{0.0015}{z - 0.9985}\right)\left(1 - \frac{0.01}{z - 0.9900}\right)}$$

$$G_c = \frac{F}{E} = \frac{0.01z - 0.01}{0.0015z - 0.0015}$$

$$(0.0015z - 0.0015)F = (0.01z - 0.01)E$$

$$\left(1 - \frac{1}{z}\right)F = \left(6.67 - \frac{6.67}{z}\right)E$$

$$f(k) = f(k-1) + 6.67e(k) - 6.67e(k-1)$$