Fuzzy Set Theory

1. Introduction

1.1 The Emergence of Fuzzy Set Theory

- From the beginning of modern science until the end of the nineteenth century, ***uncertainty*** was generally viewed as undesirable in science and the idea was to avoid it.
- From the beginning of the twentieth century ***statistical averages*** *and* ***probability theory*** have been applied to solve uncertainty of events.
- Linguistic uncertainty could not be solved until ***Lotfi A. Zadeh*** in 1965 introduced the concept of a ***fuzzy set***, a set whose boundary is not sharp, or precise.
- Fuzzy concept contrasts with the classical concept of a set, called a crisp set, whose boundary is required to be precise.

Example of crisp set: Legal Thai Coins used in 1999:

{25-satang, 50-satang, 1-baht, 5-baht, and 10-baht}

Example of fuzzy set: Cold temperature:

Temperatures are in the set to various degrees.

- Fuzzy set assigns to each member a number between 0 and 1, which indicates the degree or grade of membership in the set.
- The assignment of 0 to a particular member means that this member definitely does not belong to the set.
- The assignment of 1 means that the member definitely does belong to the set.

Applications:

- Fuzzy controllers, based on inference rules stated in natural language and represented by fuzzy sets: applied in air-conditioner, heater, rice-cooker, automobile anti-skid brake, electric train.
- Fuzzy set theory used to understand stock price fluctuations and to predict future behavior of stocks
- Fuzzy set theory used in expert systems, database and information retrieval systems, pattern recognition and clustering, signal and image processing, speech recognition, risk analysis, robotics, medicine, psychology, chemistry, ecology, and economics.

2. Fuzzy Sets: Basic Concepts and Properties

2.1 Restrictions of Classical set Theory and Logic

- In classical set theory, the boundaries of classical sets are precise.
- An individual is either definitely a member of the set or definitely not a member of it.
- In classical logic, each proposition is treated as either definitely true or definitely false.
- In many situations, it's not appropriate to make a distinction of set members by sharp transition; eg., using 25°C as a criteria to judge COLD and HOT.
- Two important laws of both classical set theory and classical logic: the law of contradiction and the law of the excluded middle, are not valid on fuzzy set and fuzzy logic.
  - The law of contradiction ($A \cap \overline{A} = \phi$): for classical logic, any proposition affirming a fact and denying it at the same time is false. For classical set, an individual cannot simultaneously be a member of a set and its complement.
  - The law of the excluded middle ($A \cup \overline{A} = X$): for classical logic, any proposition must be either true or false, but not both. For classical set, an individual must be a member of either a set or its complement.

2.2 Membership Functions

- ***Degree of membership*** of an individual in a fuzzy set expresses the ***degree of compatibility*** of the individual with the concept represented by the fuzzy set.
- Each fuzzy set, *A*, is defined in terms of a relevant universal set, *X*, by a function analogous to the characteristic function. This function, called a ***membership function***, assigns to each element *x* of *X* a number, *A*(*x*), in the closed unit interval [0, 1] that characterizes the degree of membership of *x* in *A*.

Membership functions are functions of the form

$$A : X \to [0,1] \tag{2.2-1}$$

- The universal set *X* is always assumed to be a classical set.

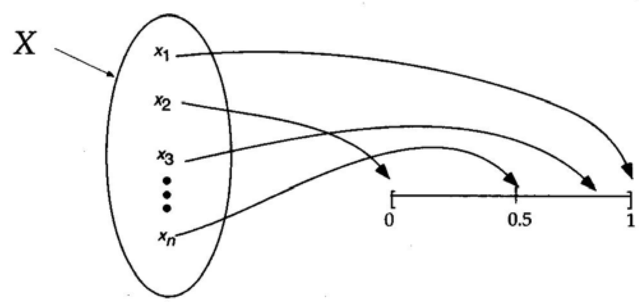

Figure 2.2-1 General form of membership functions.

- Classical sets can be viewed as special fuzzy sets whose members have 0 or 1 degree of membership.

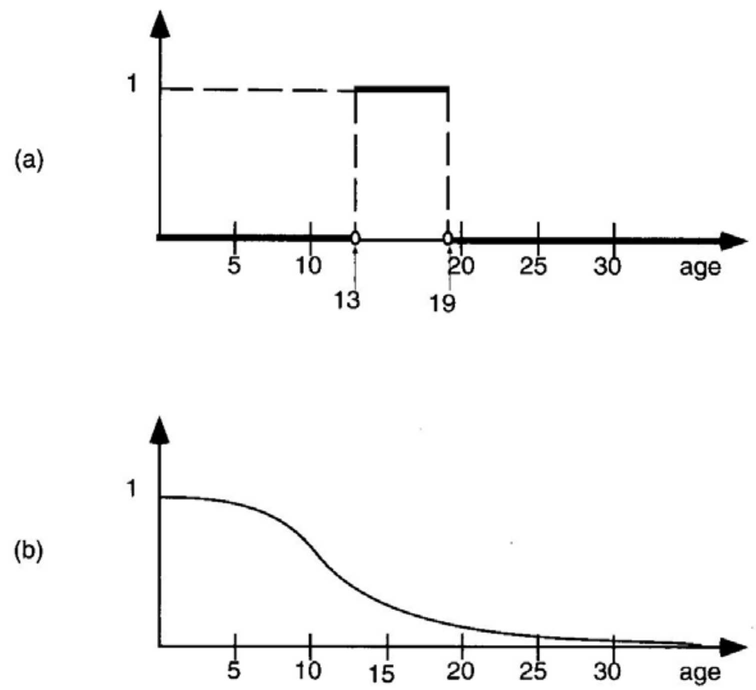

Figure 2.2-2 Membership function of (a) the set of teenagers (b) the set of young people

2.3 Representations of Membership Functions

Graphical Representation

- Applicable to membership functions whose universal sets are either the one-dimensional Euclidean space or the two-dimensional Euclidean space and either discrete or continuous.

Level Number	Educational Level Attained
0	no education
1	elementary school
2	high school
3	two-year college degree
4	bachelor's degree
5	master's degree
6	doctoral degree

Table 2.3-1 Levels of Education

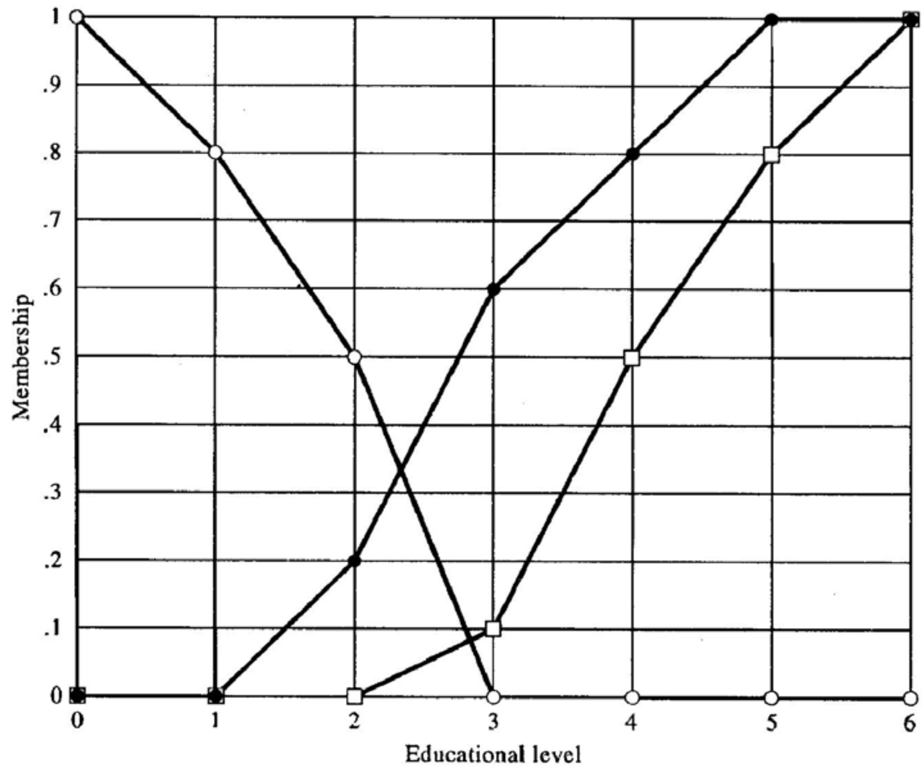

Figure 2.3-1 Examples of fuzzy sets expressing the concepts of people who are little educated (◦), highly educated (•), and very highly educated (▯).

Class Level	Credit Hours
Freshman	0–32
Sophomore	33–62
Junior	63–94
Senior	95–126

Table 2.3-2 Undergraduate class levels

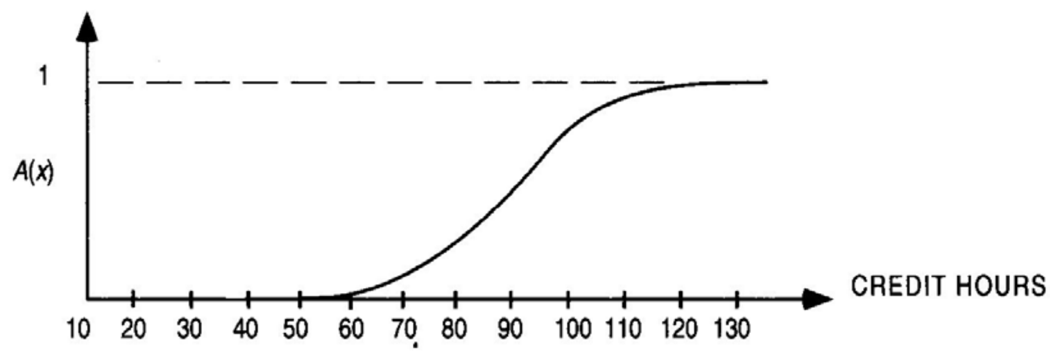

Figure 2.3-2 Representation of experienced undergraduate students.

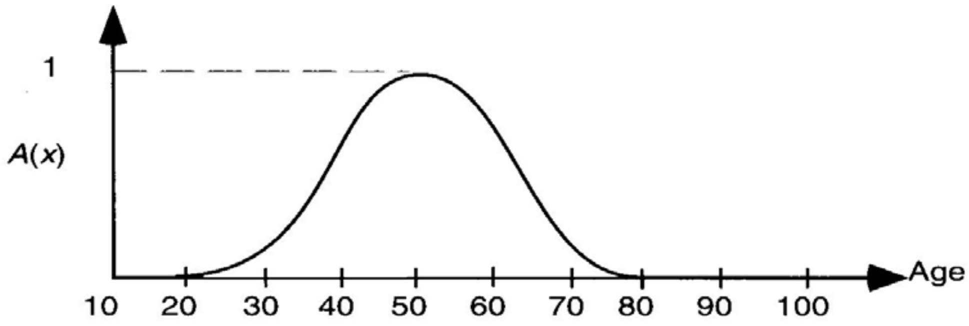

Figure 2.3-3 Middle-aged person.

Tabular and List Representations

- Applicable for universal sets that are finite.

Tabular representation of fuzzy set of very highly educated, *A*,

Education level	**Degree of Membership in *A***
No education (0)	0
Elementary school (1)	0
High school (2)	0
Two-year college degree (3)	0.1
Bachelor's degree (4)	0.5
Master's degree (5)	0.8
Doctoral degree (6)	1

List representation of fuzzy set of very highly educated, *A*,

$$A = \{< \text{No education}, 0 >, < \text{Elementary school}, 0 >, < \text{High school}, 0 >, < \text{Two-year college degree}, 0.1 >,$$
$$< \text{Bachelor's degree}, 0.5 >, < \text{Master's degree}, 0.8 >, < \text{Doctoral degree}, 1 >\} \qquad (2.3\text{-}1)$$

$$A = \{< 0, 0 >, < 1, 0 >, < 2, 0 >, < 3, 0.1 >, < 4, 0.5 >, < 5, 0.8 >, < 6, 1 >\} \qquad (2.3\text{-}2)$$

$$A = 0/0 + 0/1 + 0/2 + 0.1/3 + 0.5/4 + 0.8/5 + 1/6 \qquad (2.3\text{-}3)$$

$$A = 0.1/3 + 0.5/4 + 0.8/5 + 1/6 \qquad (2.3\text{-}4)$$

The generalized notation of list representation:

$$A = \sum A(x)/x \qquad (2.3\text{-}5)$$

Geometric Representation

- Applicable for universal sets that are finite. Universal set X: a finite set containing n elements:

$$X = \{x_1, x_2, \ldots, x_n\} \tag{2.3-6}$$

- All fuzzy sets that can be defined on a universal set with n elements can be represented by points in the ***n*-dimensional unit cube.**

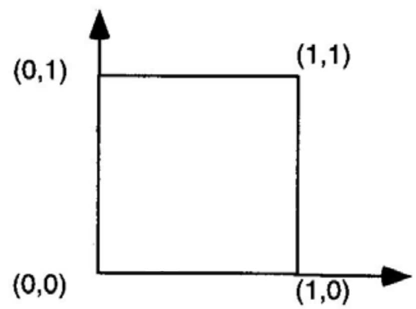

Figure 2.3-4 Two-dimensional unit cube.

- In the case of $n = 2$, there are four crisp sets: <0, 0> or empty set, <1, 1> or universal set, <1, 0> and <0, 1> locating on the vertices.
- Fuzzy sets are represented by points located anywhere within the square.

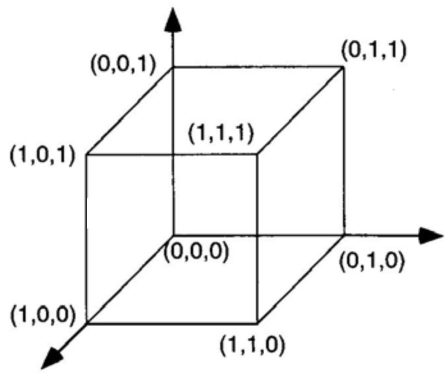

Figure 2.3-5 Three-dimensional unit cube.

- In the case of *n* = 3, there are eight crisp sets: <0, 0, 0> or empty set, <0, 0, 1>, <0, 1, 0>, <0, 1, 1>, <1, 0, 0>, <1, 0, 1>, <1, 1, 0>, as well as <1, 1, 1> or universal set.
- Every point of the unit cube represents a fuzzy set.

Analytic Representation

- Applicable for universal sets that are infinite.
- The universal set of the fuzzy set (fuzzy number) *about 6* is the set of all real numbers.

$$A(x) = \begin{cases} x-5 & \text{when } 5 \le x \le 6 \\ 7-x & \text{when } 6 \le x \le 7 \\ 0 & \text{otherwise} \end{cases} \tag{2.3-7}$$

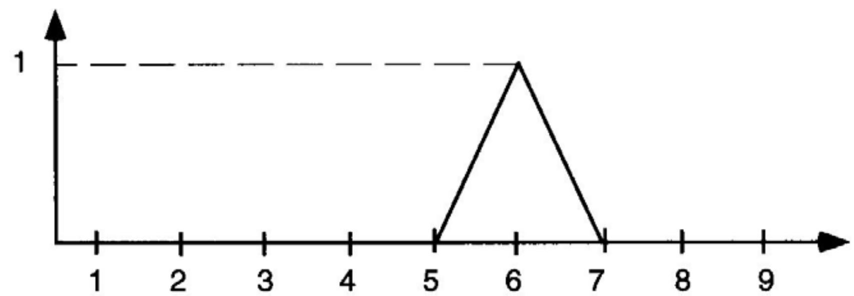

Figure 2.3-6 A membership function of *about 6*.

Triangular-shaped membership function is characterized by the three parameters, *a*, *b*, and *s*.

$$A(x) = \begin{cases} b\left(1 - \dfrac{|x-a|}{s}\right) & \text{when } a - s \le x \le a + s \\ 0 & \text{otherwise} \end{cases} \tag{2.3-8}$$

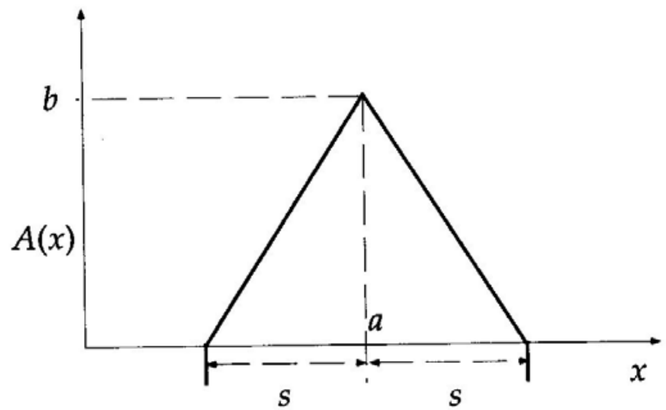

Figure 2.3-7 Generic, symmetric, and triangular membership functions.

Trapezoidal-shaped membership function is characterized by the five parameters, *a*, *b*, *c*, *d*, and *e*.

$$A(x) = \begin{cases} \dfrac{(a-x)e}{a-b} & \text{when } a \le x \le b \\ e & \text{when } b \le x \le c \\ \dfrac{(d-x)e}{d-c} & \text{when } c \le x \le d \\ 0 & \text{otherwise} \end{cases} \qquad (2.3\text{-}9)$$

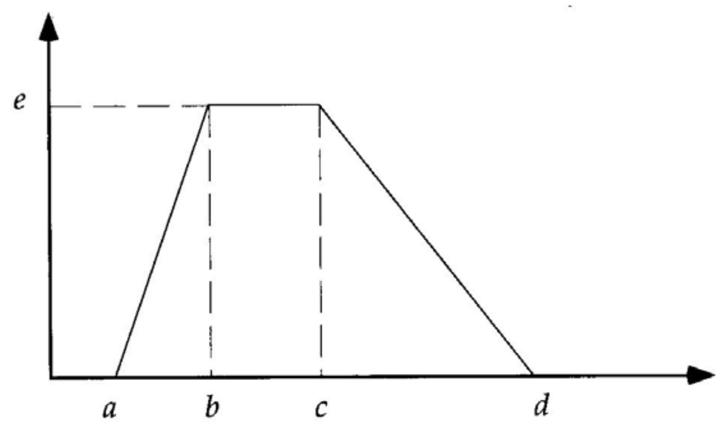

Figure 2.3-8 Trapezoidal membership function.

Bell-shaped membership function is characterized by three parameters, a, b, and c.

$$A(x) = ce^{-\frac{(x-a)^2}{b}} \tag{2.3-10}$$

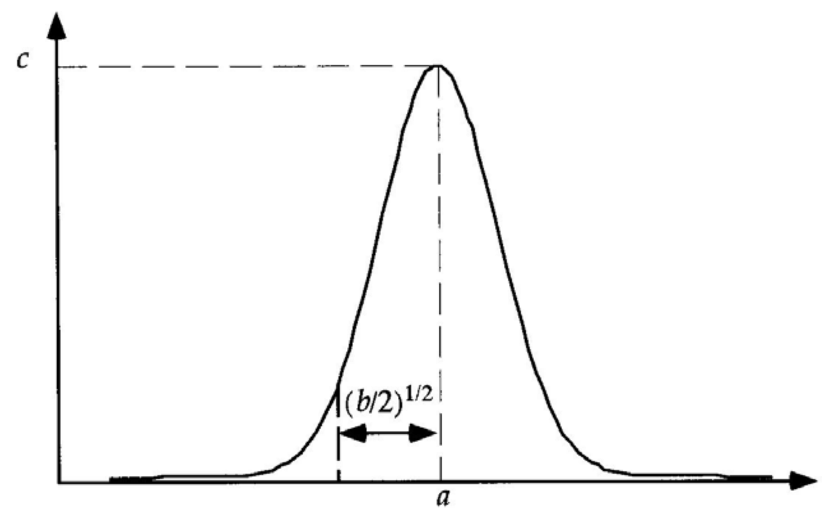

Figure 2.3-9 Bell-shaped fuzzy set.

2.4 Constructing Fuzzy Sets

- The appropriate fuzzy sets should be constructed by which the intended meanings of relevant linguistic terms are adequately captured.
- Mathematical formula is feasible for linguistic terms that are perfectly represented in the given application context by some individuals in the universal set, called ***ideal prototypes***, provided that the compatibility of other individuals with these ideal prototypes can be expressed mathematically by a meaningful ***similarity function***.

Example: membership function of a fuzzy set *E* of *almost equilateral triangles*

- According to the knowledge of geometry, an equilateral triangle has three equal angles of 60°.

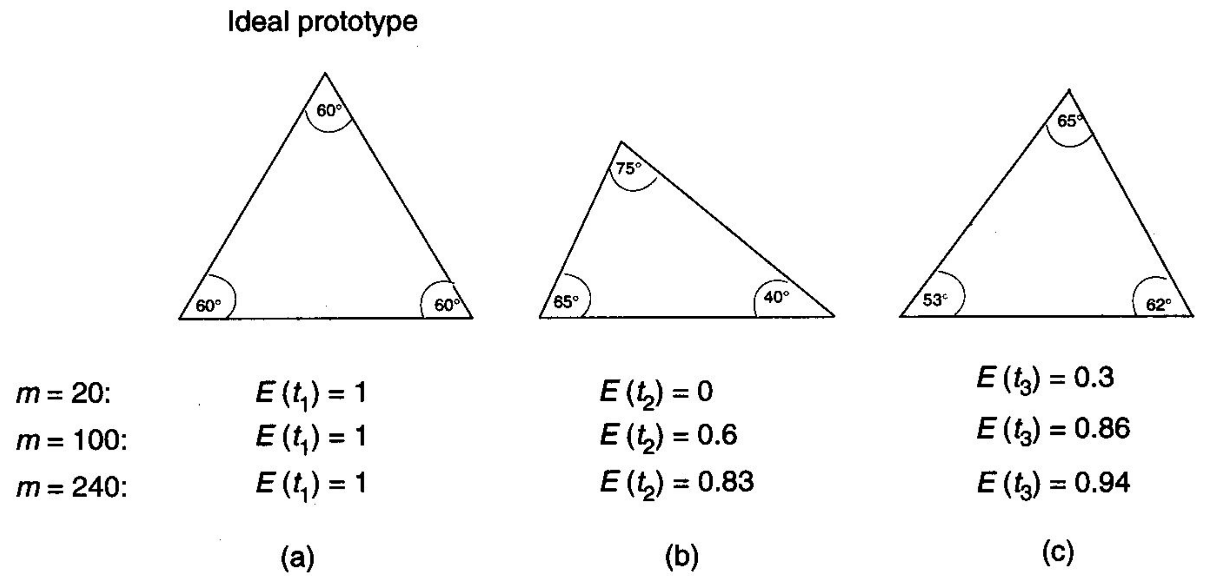

Figure 2.4-1 Membership grades of the pictured triangles in the fuzzy set of almost equilateral triangles under three distinct definitions.

$$E(t_i) = \begin{cases} 1 - \dfrac{d(t_i)}{m} & \text{when } d(t_i) \le m \\ 0 & \text{when } d(t_i) > m \end{cases} \qquad (2.4\text{-}1)$$

$d(t_i)$: the difference between each angle and 60°

m: the largest acceptable deviation of triangle t_i from the equilateral triangle t_1

- In some cases, it is reasonable to request that the expert define a membership function for a linguistic term in a given application context, either completely or to exemplify it for some selected individuals in the universal set.
- If it is not feasible to define the membership function completely, the expert exemplifies it for some representative individuals of the universal set in a set of pairs $<x, A(x)>$. This set is then used for constructing the full membership function by selecting an appropriate class of functions (triangular, trapezoidal, bell-shaped, etc.) and employ some relevant curve-fitting method to determine the function that fits best with the given samples.
- Another way is to use an appropriate neural network to construct the membership function by “learning” from the given samples.
- When several experts are employed, their opinions must be properly aggregated to determine relevant membership grades.

Example:

There are five springboard divers: Nice, Bonnie, Cathy, Dina, Eva.

There are ten referees: r_1, r_2, ..., r_{10}.

A membership function A that captures the linguistic term excellent diver.

$$A = 0.3/\text{Alice} + 0.4/\text{Bonnie} + 0.6/\text{Cathy} + 0.9/\text{Dina} + 0.6/\text{Eva} \tag{2.4-2}$$

	Alice	Bonnie	Cathy	Dina	Eva
r_1	1	1	1	1	1
r_2	0	0	1	1	1
r_3	0	1	0	1	0
r_4	1	0	1	1	1
r_5	0	0	1	1	1
r_6	0	1	1	1	1
r_7	0	0	0	0	0
r_8	1	1	1	1	1
r_9	0	0	0	1	0
r_{10}	0	0	0	1	0

Table 2.4-1 The diving survey

2.5 Operations on Fuzzy Sets

Standard Fuzzy Complement

While for each $x \in X$, $A(x)$ expresses the degree to which x belongs to A, $\overline{A}(x)$ expresses the degree to which x does not belong to A.

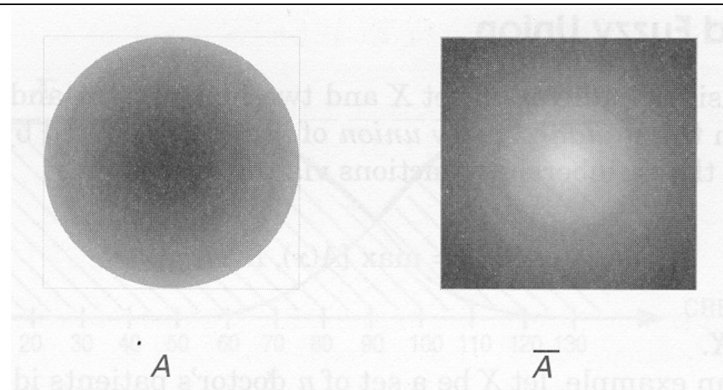

Figure 2.5-1 Fuzzy complement.

$$\overline{A}(x) = 1 - A(x) \tag{2.5-1}$$

for all $x \in X$.

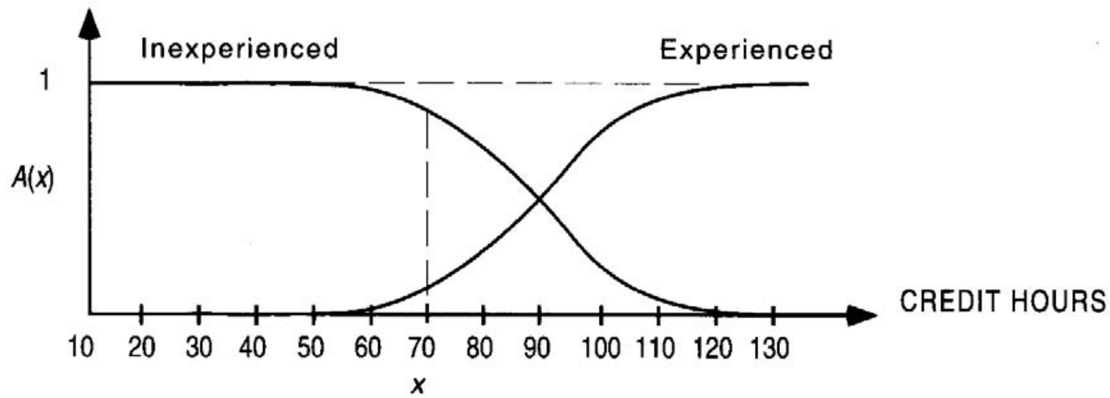

Figure 2.5-2 The set of experienced undergraduate students and its complement.

Standard Fuzzy Union

Consider a universal set *X* and two fuzzy sets *A* and *B* defined on *X*. The ***standard fuzzy union*** of *A* and *B* is denoted by $A \cup B$.

$$(A \cup B)(x) = \max[A(x), B(x)] \tag{2.5-2}$$

for all $x \in X$.

Patients	*A*=high blood pressure	*B*=high fever	$A \cup B$
1	1.0	1.0	1.0
2	0.5	0.6	0.6
3	1.0	0.1	1.0
...	...	...	...
n	0.1	0.7	0.7

Table 2.5-1 Fuzzy set of patients that have *high blood pressure or high fever*, $A \cup B$.

Te law of excluded middle,

$$A \cup \overline{A} = X \tag{2.5-3}$$

of classical set theory does not hold for fuzzy sets under the standard fuzzy union and the standard fuzzy complement.

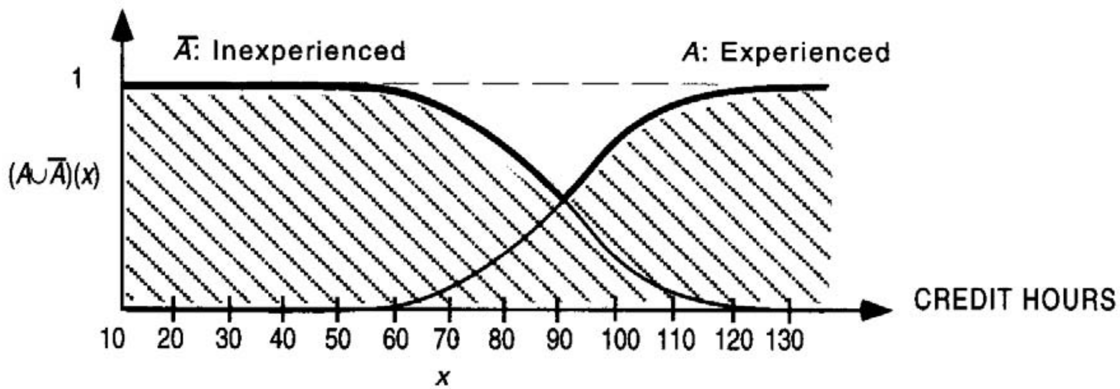

Figure 2.5-3 Union of a fuzzy set and its complement.

Standard Fuzzy Intersection

Consider again two fuzzy sets *A* and *B*, defined on *X* The ***standard fuzzy intersection*** is denoted by $A \cap B$.

$$(A \cap B)(x) = \min[A(x), B(x)] \tag{2.5-4}$$

for all $x \in X$.

River	*A* = Long River	*B* = Navigable River	$A \cap B$
Amazon	1.0	0.8	0.8
Nile	0.9	0.7	0.7
Yang-Tse	0.8	0.8	0.8
Danube	0.5	0.6	0.5
Rhine	0.4	0.3	0.3

Table 2.5-2 Fuzzy set of rivers that are *long and navigable*, $A \cap B$.

The law of contradiction

$$A \cap \overline{A} = \varnothing \tag{2.5-5}$$

of classical set theory does not hold for fuzzy sets under the standard fuzzy intersection and standard fuzzy complement.

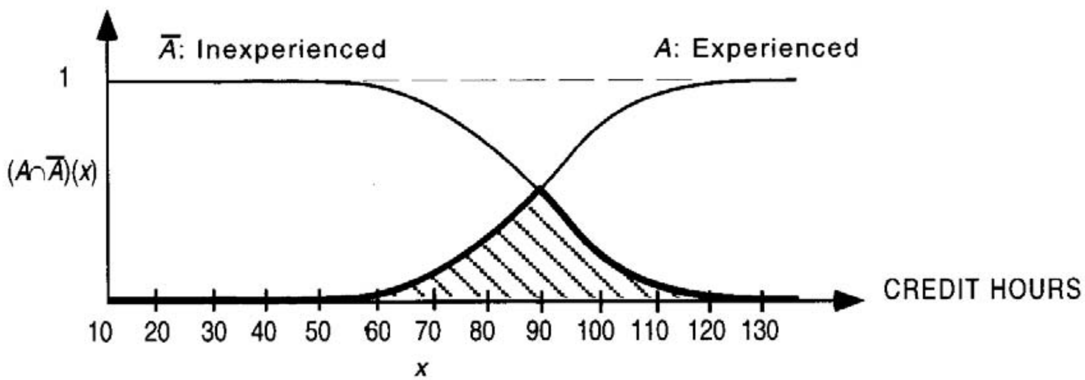

Figure 2.5-4 Intersection of a fuzzy set and its complement.

3. Fuzzy Sets: Further Properties

3.1 α-Cuts of Fuzzy Sets

$X = \{0, 10, 20, ..., 100\}$

$E = 0/0 + 0/10 + 0/20 + 0.2/30 + 0.4/40 + 0.6/50 + 0.8/60 + 1/70 + 1/80 + 1/90 + 1/100$

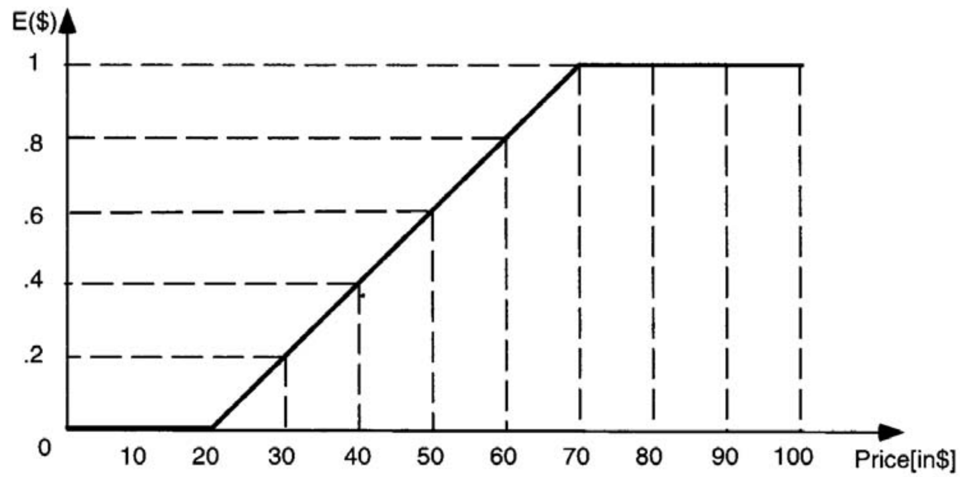

Figure 3.1-1 Fuzzy set expressing the concept of an expensive book.

Definition: For a fuzzy set A the ***α-cut*** (alpha-cut) of A,

$$^{\alpha}A = \{x \in X \mid A(x) \geq \alpha\} \tag{3.1-1}$$

for any $\alpha \in [0, 1]$.

For the continuous version of *E*,

$^{0}E = [0, 100]$, $^{0.2}E = [30, 100]$, $^{0.5}E = [45, 100]$, $^{0.9}E = [65, 100]$, $^{1}E = [70,100]$

For the discrete version of *E*,

$^{0}E = \{0, 10, ..., 100\}$, $^{0.2}E = \{30, 40, ..., 100\}$, $^{0.5}E = \{50, 60, ..., 100\}$, $^{0.9}E = {}^{1}E = \{70, 80, 90, 100\}$

For any fuzzy set A, if $\alpha_1 < \alpha_2$, then $^{\alpha 1}A \supseteq {}^{\alpha 2}A$ and, consequently,

$$^{\alpha 1}A \cap {}^{\alpha 2}A = {}^{\alpha 2}A \tag{3.1-2}$$

$$^{\alpha 1}A \cup {}^{\alpha 2}A = {}^{\alpha 1}A \tag{3.1-3}$$

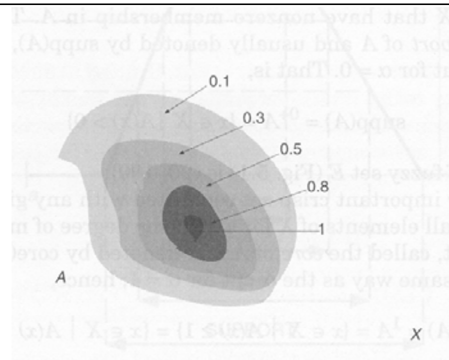

Figure 3.1-2 Nested structure of α-cuts.

Defintition: For a fuzzy set A the **strong α-cut**, ${}^{\alpha+}A$,

$$ {}^{\alpha+}A = \{x \in X \mid A(x) > \alpha\} \tag{3.1-4} $$

For the continuous version of *E*,

${}^{0+}E = (20, 100]$, ${}^{0.2+}E = (30, 100]$, ${}^{0.5+}E = (45, 100]$, ${}^{0.9+}E = (65, 100]$, ${}^{1+}E = \varnothing$

For the discrete version of *E*,

${}^{0+}E = \{30, 40, ..., 100\}$, ${}^{0.2+}E = \{40, 50, ..., 100\}$, ${}^{0.5+}E = \{50, 60, ..., 100\}$, ${}^{0.9+}E = \{70, 80, 90, 100\}$, ${}^{1+}E = \varnothing$

If $\alpha_1 < \alpha_2$, then ${}^{\alpha 1+}A \supseteq {}^{\alpha 2+}A$ and, consequently,

$$ {}^{\alpha 1+}A \cap {}^{\alpha 2+}A = {}^{\alpha 2+}A \tag{3.1-5} $$

$$ {}^{\alpha 1+}A \cup {}^{\alpha 2+}A = {}^{\alpha 1+}A \tag{3.1-6} $$

3.1.1 Support, Core, and Height

Defintition: The set of all elements of the universal set X that have nonzero membership in A is called ***the support of A***,

$$\text{supp}(A) = {}^{0+}A = \{x \in X \mid A(x) > 0\} \tag{3.1.1-1}$$

The support of fuzzy set E is (20, 100].

Defintition: The set of all elements of X for which the degree of membership in A is 1 is called ***the core of A***,

$$\text{core}(A) = {}^{1}A = \{x \in X \mid A(x) \geq 1\} = \{x \in X \mid A(x) = 1\} \tag{3.1.1-2}$$

The core of set E is [70, 100].

Defintition: The largest value of α for which the α-cut is not empty is called ***the height*** of a fuzzy set.

When $h(A) = 1$ set A is called ***normal***; otherwise it is called ***subnormal***.

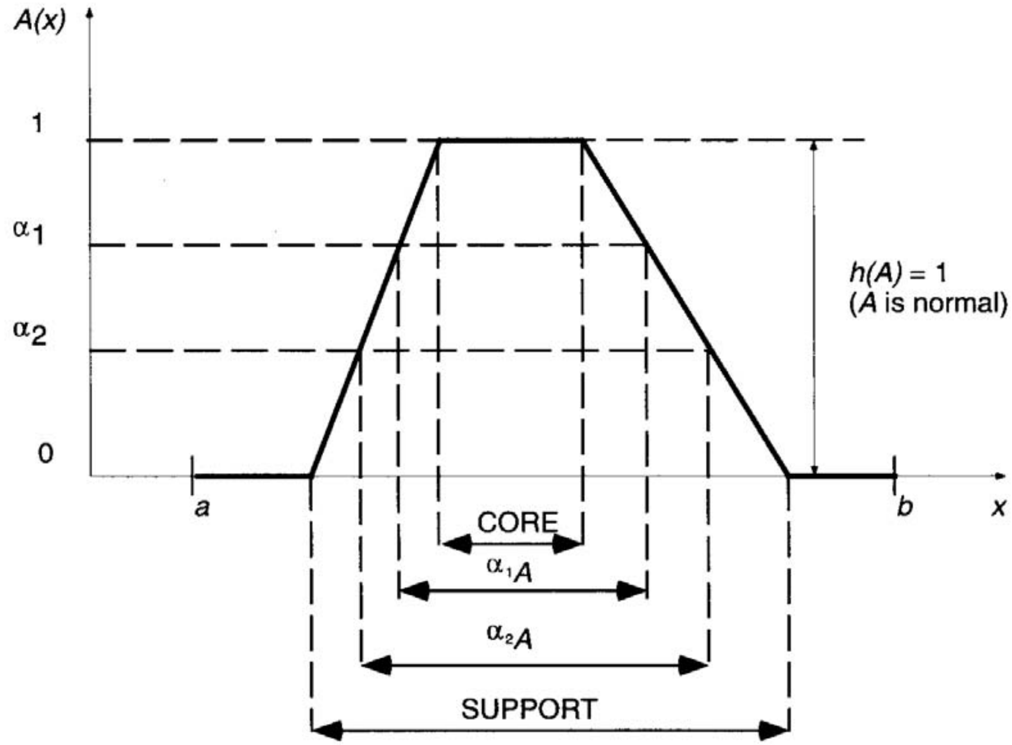

Figure 3.1.1-1 Illustration of the concepts of support, core, height, and α-cuts.

3.1.2 Level Set

Defintion: ***The level set of A***, $L(A)$,

$$L(A) = \{\alpha \in [0,1] \mid A(x) = \alpha \text{ for some } x \in X\} \tag{3.1.2-1}$$

The level set of the continuous version of fuzzy set E,

$$L(E) = [0, 1] \tag{3.1.2-2}$$

The level set of the discrete version of E,

$$L(E) = \{0, 0.2, 0.4, 0.6, 0.8, 1\} \tag{3.1.2-3}$$

3.2 α-Cut Presentation

$$A = 0.2/x_1+0.4/x_2+0.6/x_3+0.8/x_4+1/x_5 \tag{3.2-1}$$

$$L(A) = \{0.2, 0.4, 0.6, 0.8, 1\} \tag{3.2-2}$$

$$^{0.2}A = 1/x_1+1/x_2+1/x_3+1/x_4+1/x_5 \tag{3.2-3}$$

$$^{0.4}A = 0/x_1+1/x_2+1/x_3+1/x_4+1/x_5 \tag{3.2-4}$$

$$^{0.6}A = 0/x_1+0/x_2+1/x_3+1/x_4+1/x_5 \tag{3.2-5}$$

$$^{0.8}A = 0/x_1+0/x_2+0/x_3+1/x_4+1/x_5 \tag{3.2-6}$$

$$^{1}A = 0/x_1+0/x_2+0/x_3+0/x_4+1/x_5 \tag{3.2-7}$$

$$_{\alpha}A(x) = \alpha(^{\alpha}A(x)) \tag{3.2-8}$$

$$_{0.2}A = 0.2/x_1+0.2/x_2+0.2/x_3+0.2/x_4+0.2/x_5 \tag{3.2-9}$$

$$_{0.4}A = 0/x_1+0.4/x_2+0.4/x_3+0.4/x_4+0.4/x_5 \tag{3.2-10}$$

$$_{0.6}A = 0/x_1+0/x_2+0.6/x_3+0.6/x_4+0.6/x_5 \tag{3.2-11}$$

$$_{0.8}A = 0/x_1+0/x_2+0/x_3+0.8/x_4+0.8/x_5 \tag{3.2-12}$$

$$_{1}A = 0/x_1+0/x_2+0/x_3+0/x_4+1/x_5 \tag{3.2-13}$$

$$A = {}_{0.2}A \cup {}_{0.4}A \cup {}_{0.6}A \cup {}_{0.8}A \cup {}_{1}A \tag{3.2-14}$$

Theorem 3.2-1: ***Decomposition theorem of fuzzy sets***, For any $A \in F(X)$,

$$A = \bigcup_{\alpha\in[0,1]} {}_{\alpha}A \tag{3.2-14}$$

3.3 Cutworthy Properties of Fuzzy Sets

- A property of classical sets is extended to fuzzy sets via the α-cut representation by requiring that the property be satisfied (in the classical sense) in all α-cuts of the relevant fuzzy sets.
- Any property of fuzzy sets that is derived from classical set theory is called a ***cutworthy property***.

The usual definition of this equality,

$$A = B \text{ if and only if } A(x) = B(x) \text{ for all } x \in X \tag{3.3-1}$$

The equivalent definition of equality based on the α-cut representations of the fuzzy sets,

$$A = B \text{ if and only if } {}^{\alpha}A = {}^{\alpha}B \text{ for all } \alpha \in [0, 1] \tag{3.3-2}$$

$A \subseteq B$ if and only if ${}^{\alpha}A \subseteq {}^{\alpha}B$ for any $\alpha \in [0, 1]$.

Theorem 3.3-1. For any two fuzzy sets A, B and $\alpha \in [0, 1]$,

$${}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B \tag{3.3-3}$$

$${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B \tag{3.3-4}$$

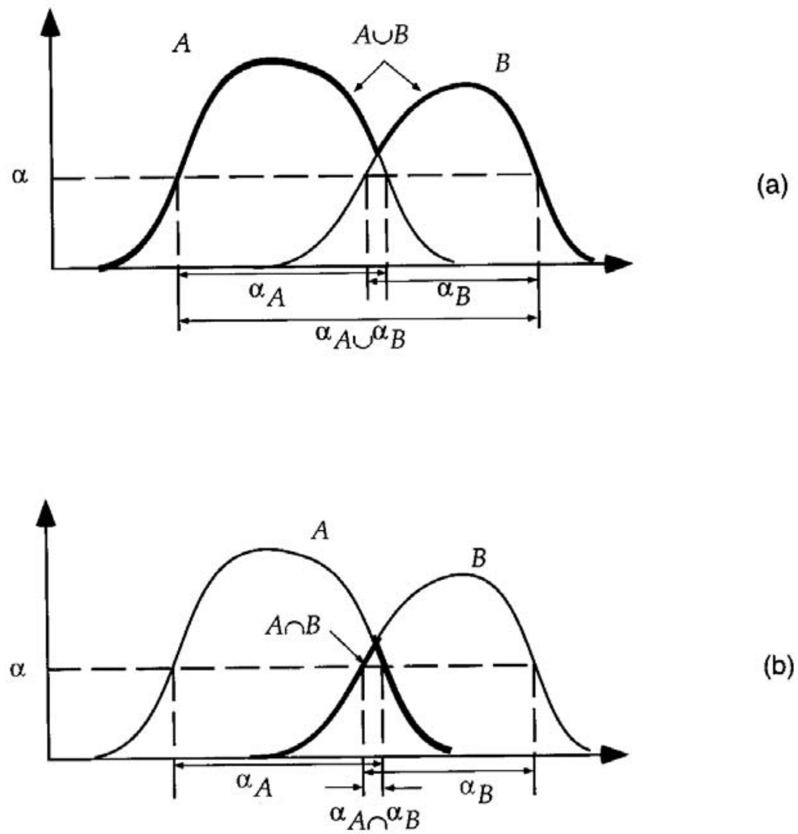

Figure 3.3-1 Illustration of (a) union (b) intersection.

$$A = 0.2/x_1+0.4/x_2+0.6/x_3+0.8/x_4+1/x_5 \tag{3.3-5}$$

$$B = 1/x_1+0.7/x_2+0.5/x_3+0.3/x_4+0.1/x_5 \tag{3.3-6}$$

$$A\cup B = 1/x_1+0.7/x_2+0.6/x_3+0.8/x_4+1/x_5 \tag{3.3-7}$$

$$A\cap B = 0.2/x_1+0.4/x_2+0.5/x_3+0.3/x_4+0.1/x_5 \tag{3.3-8}$$

$\alpha = 0.5$,

$$^{\alpha}A = \{x_3, x_4, x_5\} \tag{3.3-9}$$

$$^{\alpha}B = \{x_1, x_2, x_3\} \tag{3.3-10}$$

$$^{\alpha}(A\cup B) = \{x_1, x_2, x_3, x_4, x_5\} \tag{3.3-11}$$

$$^{\alpha}(A\cap B) = \{x_3\} \tag{3.3-12}$$

$$^{\alpha}A \cup {}^{\alpha}B = \{x_1, x_2, x_3, x_4, x_5\} = {}^{\alpha}(A\cup B) \tag{3.3-13}$$

$$^{\alpha}A \cap {}^{\alpha}B = \{x_3\} = {}^{\alpha}(A\cap B) \tag{3.3-14}$$

Definition: ***The convexity of fuzzy sets***, A fuzzy set defined on the set of real numbers (or, more generally, on any *n*-dimensional Euclidean space) is said to be convex if and only if all of its α-cuts are convex in the classical sense.

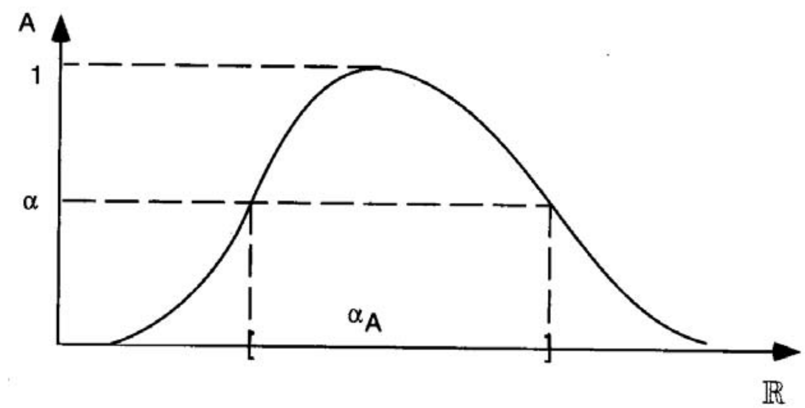

Figure 3.3-2 A convex fuzzy set of real numbers.

- For a fuzzy set to be convex the graph must have just one peak.
- All α-cuts of convex fuzzy set are closed intervals of real.

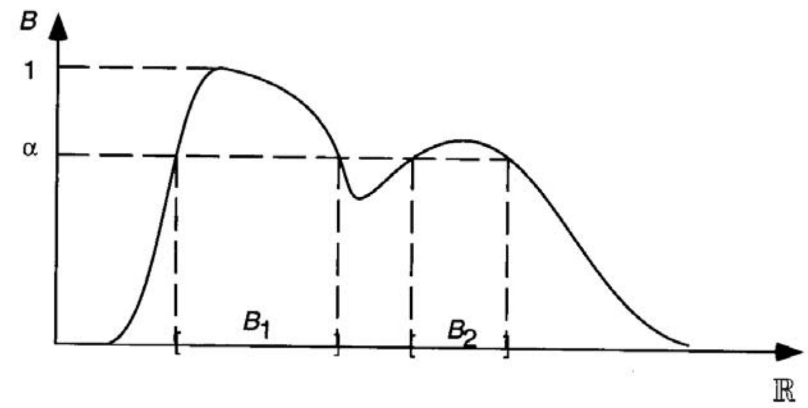

Figure 3.3-3 Nonconvex fuzzy set (${}^{\alpha}B = B_1 \cup B_2$).

3.4 Extension Principle

Age (in years)	20	25	30	35	40	45	50	55	60	65
Salary ($ in K)	2.5	2.5	3.0	3.5	3.5	4.0	4.0	4.5	4.5	5.0

Table 3.4-1 Employees and their salaries.

Age, X = {20, 25, 30, 35, 40, 45, 50, 55, 60, 65}

Salary, Y = {2.5, 3, 3.5, 4, 4.5, 5}

Fuzzy set of young age, $A(x)$, $A(x)$ =1/20+1/25+0.8/30+0.6/35+0.4/40+0.2/45+0/50+0/55+0/60+0/65

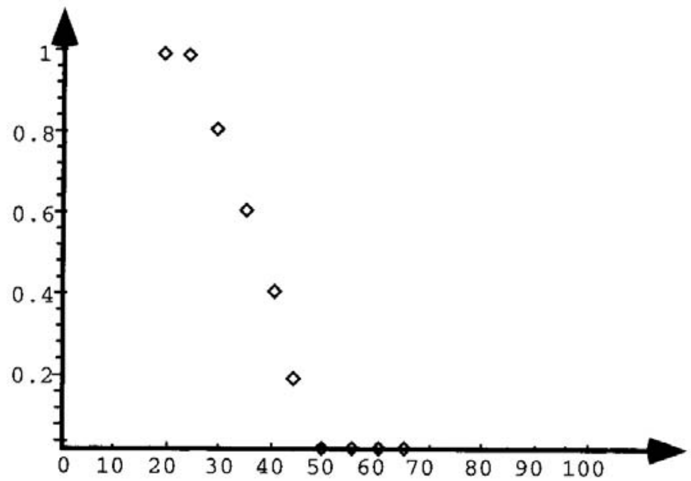

Figure 3.4-1 Fuzzy set employed to express the concept of a young employee

Fuzzy set of young employee's salary,

$$A(y) = A(x)/f(x) \tag{3.4-1}$$

$$A(y) = 1/f(20)+1/f(25)+0.8/f(30)+0.6/f(35)+0.4/f(40)+0.2/f(45)+0/f(50)+0/f(55)+0/f(60)+0/f(65)$$

$$= 1/2.5+1/2.5+0.8/3+0.6/3.5+0.4/3.5+0.2/4+0/4+0/4.5+0/4.5+0/5 \tag{3.4-2}$$

$$A(y) = 1/2.5 + 0.8/3 + 0.6/3.5 + 0.2/4 + 0/4.5 + 0/5 \tag{3.4-3}$$

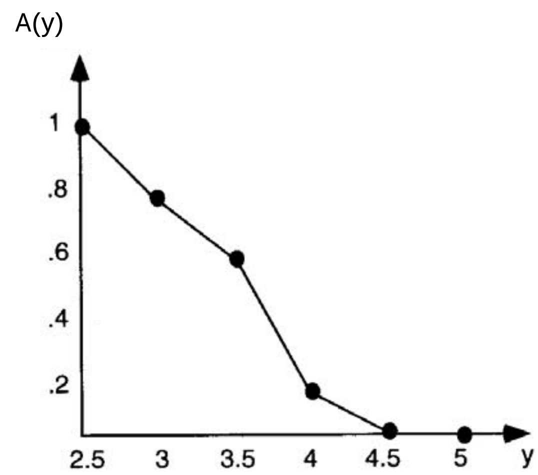

Figure 3.4-2 Fuzzy set expressing the concept of the salary of young employees.

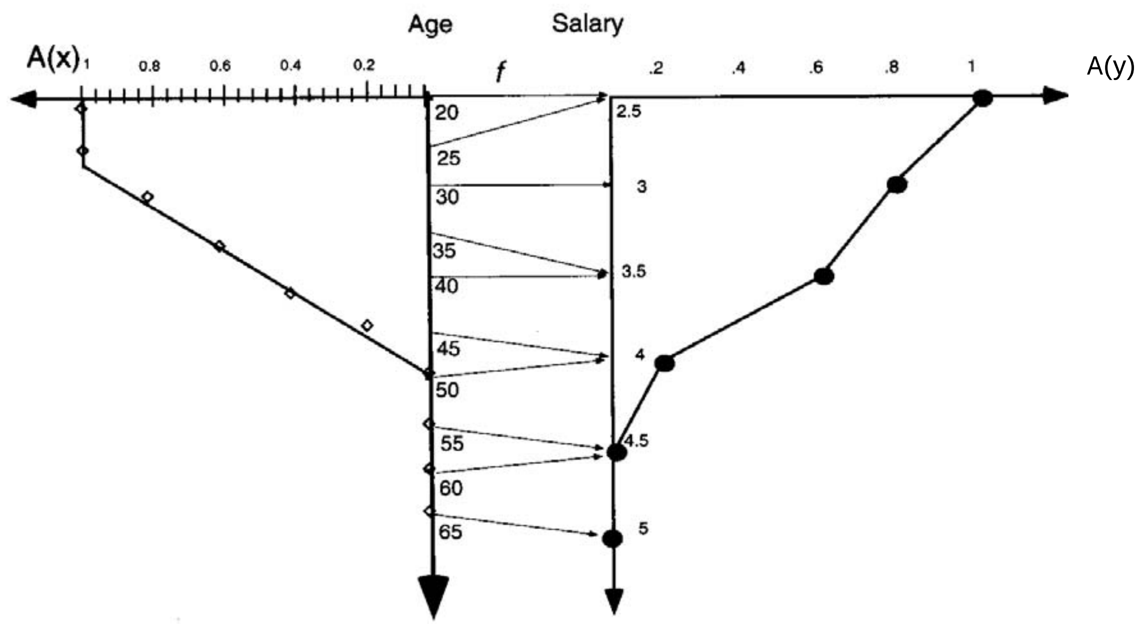

Figure 3.4-3 Illustration of the extension principle.

Extension Principle. Let f: $X \to Y$ where X and Y denote finite crisp sets, be a given function. Two functions may be induced from f. One, denoted by $\tilde{f}$, is a function from $F(X)$ to $F(Y)$. The other, denoted by $\tilde{f}^{-1}$, is a function from $F(Y)$ to $F(X)$. These functions are defined by

$$[\tilde{f}(A)](y) = \max_{x|y=f(x)} A(x) \tag{3.4-4}$$

for any $A \in F(X)$, $y \in Y$

$$[\tilde{f}^{-1}(B)](x) = B(f(x)) \tag{3.4-5}$$

for any $B \in F(X=Y)$, and $x \in X$.

When f is a continuous function defined on the set of real numbers,

$$[\tilde{f}(A)](y) = \sup_{x|y=f(x)} A(x) \tag{3.4-6}$$

where sup denotes the supremum (sup A is the maximum number in A).

Fuzzy set of a low salary, $B(y)$,

$$B(y) = 1/2.5 + 0.75/3 + 0.5/3.5 + 0.25/4 + 0/4.5 + 0/5 \tag{3.4-7}$$

Fuzzy set of low salary respect to the age of employees,

$$\begin{aligned}\tilde{f}^{-1}(B) &= B(f(20))/20 + B(f(25))/25 + B(f(30))/30 + B(f(35))/35 + B(f(40))/40 + B(f(45))/45 \\ &\quad + B(f(50))/50 + B(f(55))/55 + B(f(60))/6 + B(f(65))/65 \\ &= 1/20 + 1/25 + 0.75/30 + 0.5/35 + 0.5/40 + 0.25/45 + 0.25/50 + 0/55 + 0/60 + 0/65\end{aligned} \tag{3.4-8}$$

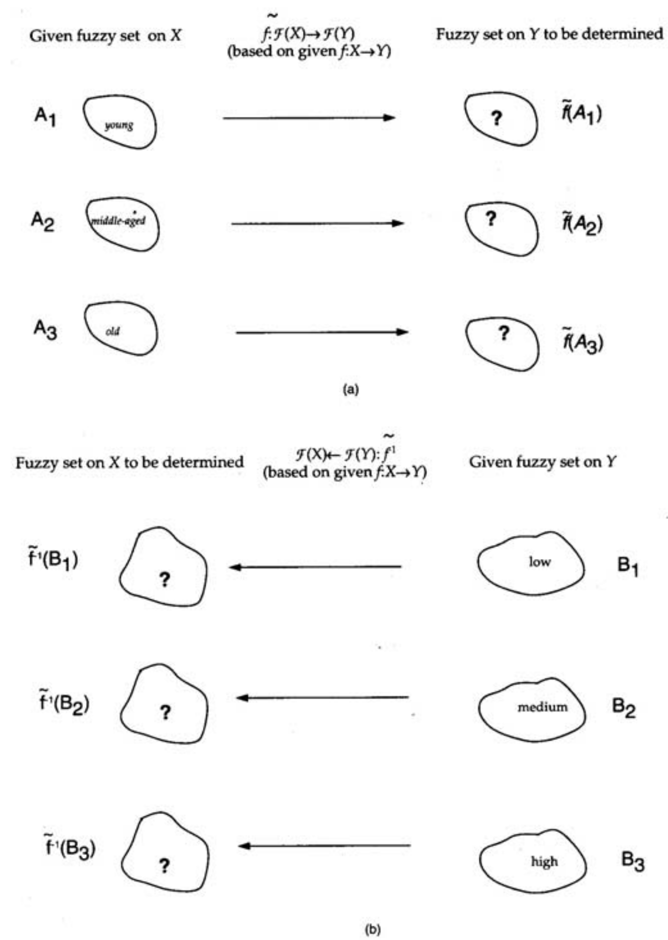

Figure 3.4-4 Extension Principle

3.5 Measurement of Fuzziness

When X is finite, fuzziness, $f(A)$,

$$f(A) = \sum_{x \in X} (1 - |2A(x) - 1|) / N(x) \tag{3.5-1}$$

$N(x)$: the number of members in Fuzzy set A

$$0 \le f(A) \le 1 \tag{3.5-2}$$

$$A(x) = 1/20+1/25+0.8/30+0.6/35+0.4/40+0.2/45+0/50+0/55+0/60+0/65 \tag{3.5-3}$$

$$f(A) = (10 - 1 - 1 - 0.6 - 0.2 - 0.2 - 0.6 - 1 - 1 - 1 - 1)/10 = 0.24 \tag{3.5-4}$$

$$B(y) = 1/2.5 + 0.75/3 + 0.5/3.5 + 0.25/4 + 0/4.5 + 0/5 \tag{3.5-5}$$

$$f(B) = (6 - 1 - 0.5 - 0 - 0.5 - 1 - 1)/6 = 0.33 \tag{3.5-6}$$

4. Fuzzy Relations

4.1 Introduction

- Classical relations describe solely the presence or absence of association (interaction, connection, etc.) between elements of two or more sets.
- Fuzzy relations are capable of capturing the strength of association (interaction, connection).
- ***Fuzzy relations*** are fuzzy sets defined on universal sets which are Cartesian products.

Example: "*x* is approximately equal to *y*" or "*x* is close to *y*,"

$$E(x, y) = \max\left(0,\ 1 - \frac{|x - y|}{c}\right) \tag{4.1-1}$$

c: a positive real number whose value is chosen in the context of each application with the aim to make the fuzzy relation *E* a good representation of the concept in that application.

The ranges of the variables are $[x_1, x_2]$ and $[y_1, y_2]$.

The universal set on which this fuzzy relation is defined is thus the Cartesian product $[x_1, x_2]\times[y_1, y_2]$.

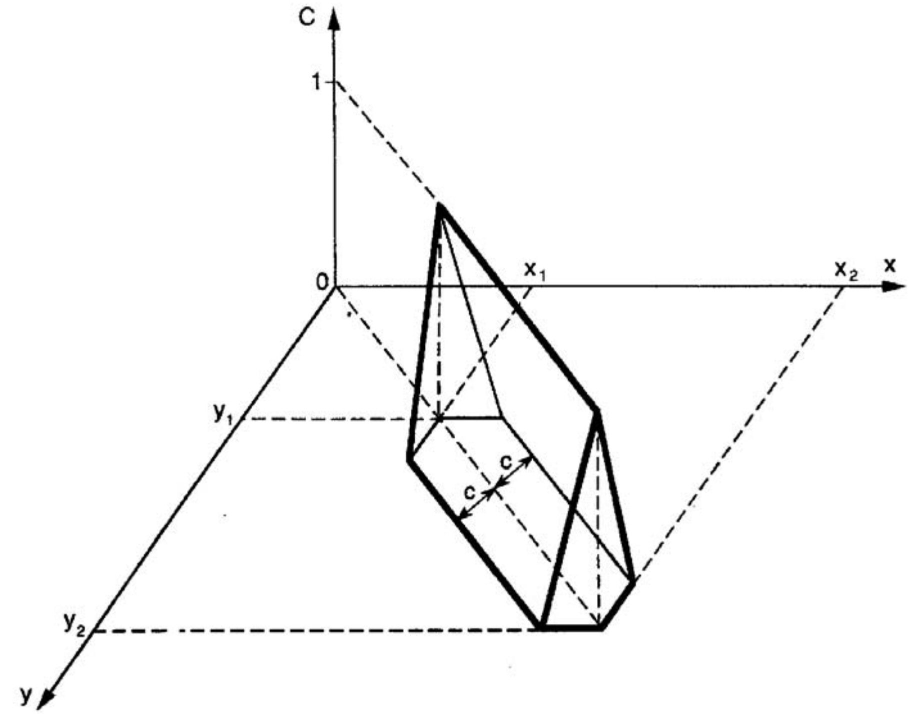

Figure 4.1-1 Membership function of the fuzzy relation “*y* is approximately equal to *x*.”

4.2 Representations

- A finite fuzzy relation can always be represented by a **list of ordered pairs (or *n*-tuples)** of the relevant Cartesian products with their membership grades.
- Pairs (or n-tuples) whose membership grades are zero are usually omitted.
- When a fuzzy relation is not finite, the membership function can be defined by a suitable **formula**.

Matrices

For a binary fuzzy relation R on $X \times Y$ where $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_m\}$.

Matrix representation **R** of the relation:

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nm} \end{bmatrix} \tag{4.2-1}$$

where $r_{ij} = R(x_i, y_j)$ for each $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$.

Example: X: a set of eight major cities,

$$X = (\text{Beijing, Chicago, London, Moscow, New York, Paris, Sydney, Tokyo}) \tag{4.2-2}$$

R: a fuzzy relation on X of the relational concept *very far from*.

R	**B**	**C**	**L**	**M**	**N**	**P**	**S**	**T**
B	0	1	0.7	0.5	1	0.7	0.6	0.1
C	1	0	0.5	0.9	0	0.5	1	1
L	0.7	0.5	0	0.3	0.5	0	1	0.7
M	0.5	0.9	0.3	0	0.9	0.3	0.8	0.5
N	1	0	0.5	0.9	0	0.5	1	1
P	0.7	0.5	0	0.3	0.5	0	1	0.7
S	0.6	1	1	0.8	1	1	0	0.6
T	1	1	0.7	0.5	1	0.7	0.6	0

- To represent a three-dimensional relation, a sequence of matrices (referred to as a three-dimensional array) is needed.
- While two of the three dimensions are represented by columns and rows of the matrices, elements of the third dimension are represented by the distinct matrices.
- For the Cartesian product $X_3 \times X_2 \times X_1$, elements of X_1 correspond to columns in the matrices, elements of X_2 correspond to rows in the matrices, and elements of X_3 correspond to distinct matrices.
- Using the same principle, arrays for higher dimensional relations are constructed.

Mappings

Example: A set of documents $D = \{d_1, d_2, ..., d_5\}$, and a set of key terms $T = \{t_1, t_2, t_3, t_4\}$.

A fuzzy relation expressing the degree of relevance of each document to each key term,

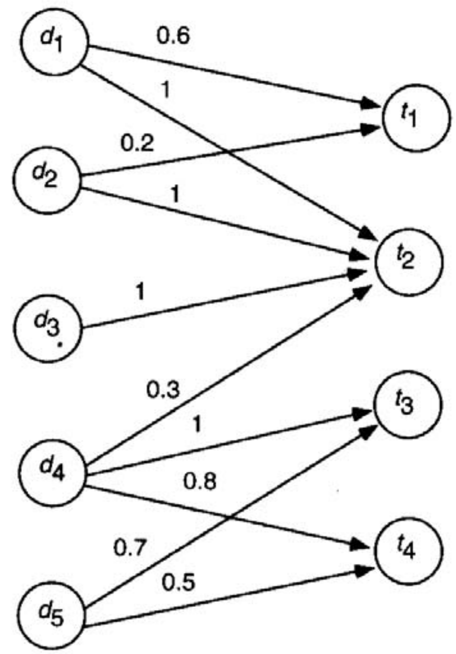

Figure 4.2-1 Representation of a fuzzy relation by the mapping diagram.

Directed Graphs

The graphical representation of a fuzzy relation on X, where $X = \{x_1, x_2, x_3, x_4\}$,

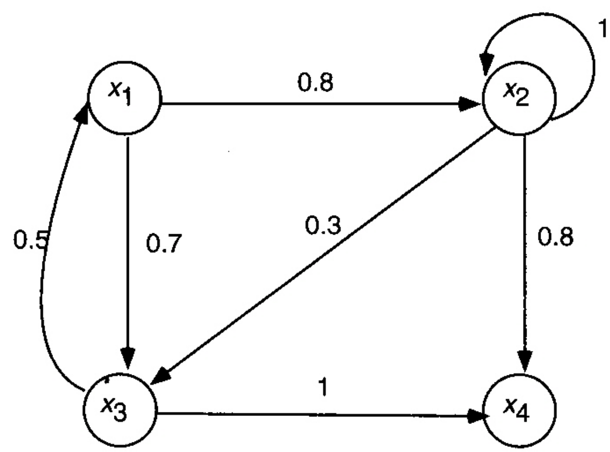

Figure 4.2-2 Representation of a fuzzy relation by a directed graph.

4.3 Operations on Binary Fuzzy Relations

- All operations and concepts on fuzzy sets; complements, intersections, unions, subset, convexity, α-cuts, and etc., are applicable to fuzzy relations.

Definition: The ***inverse*** of a fuzzy binary relation R on $X\times Y$ which is usually denoted by R^{-1}, is a relation on $Y\times X$ defined as

$$R^{-1}(y,x) = R(x,y) \tag{4.3-1}$$

for all pairs $\langle y, x\rangle \in Y\times X$.

$$(R^{-1})^{-1} = R \tag{4.3-2}$$

When R is represented by a matrix, the matrix representation of R^{-1} is obtained by exchanging the rows of the given matrix with the columns. The resulting matrix is called the ***transpose*** of the given matrix.

$$\mathbf{R} = \begin{bmatrix} 0.6 & 1 & 0 & 0 \\ 0.2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.3 & 1 & 0.8 \\ 0 & 0 & 0.7 & 0.5 \end{bmatrix} \text{ and } \mathbf{R}^{-1} = \begin{bmatrix} 0.6 & 0.2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0.3 & 0 \\ 0 & 0 & 0 & 1 & 0.7 \\ 0 & 0 & 0 & 0.8 & 0.5 \end{bmatrix} \tag{4.3-3}$$

Definition: The composition of fuzzy relations P and Q is defined for each pair $\langle x, z\rangle \in X \times Z$ by the formula

$$R(x,z) = (P \circ Q)(x,z) = \max_{y \in Y} \min[P(x,y), Q(y,z)] \tag{4.3-4}$$

Example:

$$X = \{a, b, c\},\ Y = \{1, 2, 3, 4\}, \text{ and } Z = \{A, B, C\} \tag{4.3-5}$$

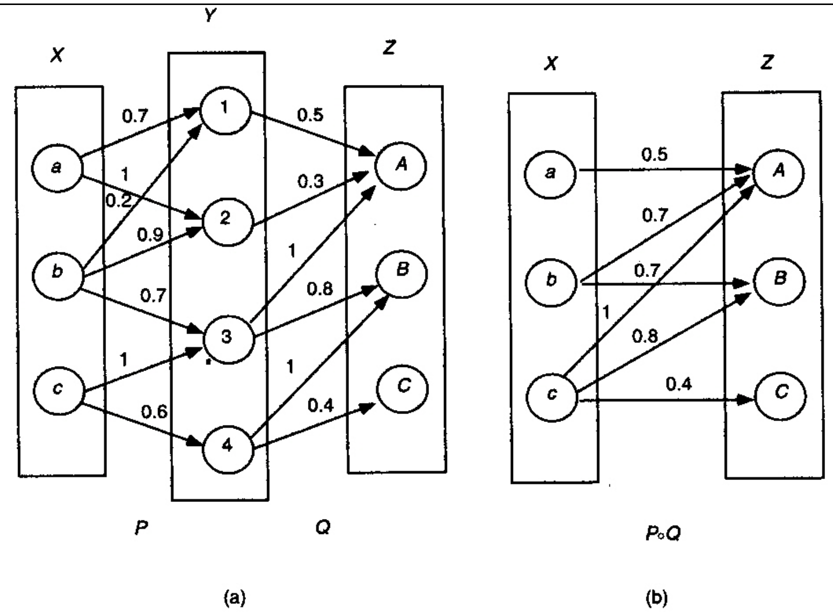

Figure 4.3-1 Illustration of the composition of binary fuzzy relations P and Q.

$$(P \circ Q)(b, A) = \max\{\min[P(b,1), Q(1, A)], \min[P(b,2), Q(2, A)], \min[P(b,3), Q(3, A)]\}$$

$$= \max\{\min[0.2,0.5], \min[0.9,0.3], \min[0.7,1]\} = \max\{0.2,0.3,0.7\} = 0.7 \quad (4.3\text{-}6)$$

Example:

$$X = \{p_1, p_2, p_3, p_4\} = \text{set of patients} \quad (4.3\text{-}7)$$

$$Y = \{s_1, s_2, s_3\} = \text{set of symptoms} \quad (4.3\text{-}8)$$

$$Z = \{d_1, d_2, d_3, d_4, d_5\} = \text{set of diseases} \quad (4.3\text{-}9)$$

A fuzzy relation P on $X \times Y$,

$$\mathbf{P} = \begin{bmatrix} 0 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.3 \\ 0.8 & 0 & 0 \\ 0.7 & 0.7 & 0.9 \end{bmatrix} \quad (4.3\text{-}10)$$

This relation describes how strongly the symptoms are manifested in the patients.

Another fuzzy relation Q on $Y \times Z$,

$$\mathbf{Q} = \begin{bmatrix} 0.7 & 0 & 0 & 0.3 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 & 0 \\ 0 & 0.7 & 0.2 & 0.9 & 0 \end{bmatrix} \quad (4.3\text{-}11)$$

This relation describes a segment of medical knowledge expressing how strongly each symptom is associated with a disease.

By performing the composition $P \circ Q = R$,

$$\mathbf{R} = \begin{bmatrix} 0 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.3 \\ 0.8 & 0 & 0 \\ 0.7 & 0.7 & 0.9 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0 & 0 & 0.3 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 & 0 \\ 0 & 0.7 & 0.2 & 0.9 & 0 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.4 & 0.3 & 0.4 & 0 \\ 0.5 & 0.5 & 0.5 & 0.4 & 0.2 \\ 0.7 & 0 & 0 & 0.3 & 0.6 \\ 0.7 & 0.7 & 0.7 & 0.9 & 0.6 \end{bmatrix} \tag{4.3-12}$$

The composite relation **R** expresses the association between patients and diseases and, hence, facilitates medical diagnosis.

$$(P \circ Q) \circ R = P \circ (Q \circ R) \tag{4.3-13}$$

$$(P \circ Q)^{-1} = Q^{-1} \circ P^{-1} \tag{4.3-14}$$

$$P \circ Q \neq Q \circ P \tag{4.3-15}$$

4.4 Fuzzy Equivalence Relations and Compatibility Relations

Definition: A fuzzy relation R on X is **reflexive** if and only if $R(x, x) = 1$ for all $x \in X$.

Definition: A fuzzy relation R is **symmetric** if and only if $R(x, y) = R(y, x)$ for all $x, y \in X$.

Definition: A fuzzy relation R is **transitive** if and only if

$$R(x,z) \geq \max_{y \in Y} \min[R(x,y), R(y,z)] \tag{4.4-1}$$

for all $x, z \in X$.

Definition: A fuzzy relation on X that is reflexive, symmetric, and transitive is a **fuzzy equivalence relation**.

Example:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0.8 & 0 & 0.8 & 0.5 & 0 \\ 0.8 & 1 & 0 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.8 \\ 0.8 & 1 & 0 & 1 & 0.5 & 0 \\ 0.5 & 0.5 & 0 & 0.5 & 1 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 1 \end{bmatrix} \tag{4.4-2}$$

- The fuzzy relation Q is reflexive, symmetric, and transitive.
- Each α-cut of this fuzzy relation is a crisp equivalence relation and the partitions become more refined when the value of α is increased.

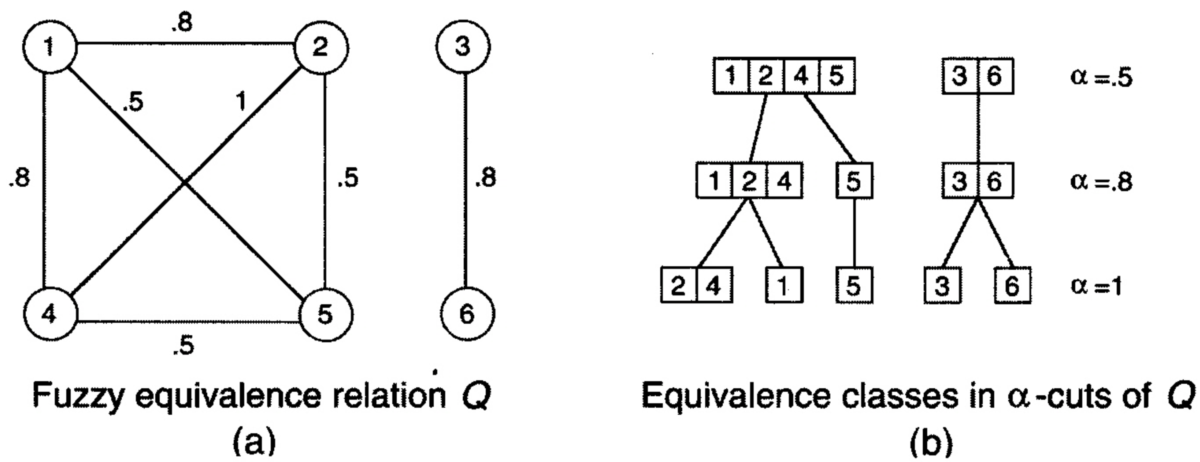

Figure 4.4-1 An example of fuzzy equivalence relation.

Definition: **Fuzzy compatibility relation** is a fuzzy relation which is reflexive and symmetric but not transitive.

Example:

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 0 & 0.8 & 0.9 & 0 \\ 1 & 1 & 0.8 & 0.9 & 0.5 & 0 \\ 0 & 0.8 & 1 & 0 & 0 & 0.8 \\ 0.8 & 0.9 & 0 & 1 & 1 & 0 \\ 0.9 & 0.5 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 1 \end{bmatrix} \tag{4.4-3}$$

The fuzzy relation R is reflexive and symmetric, but not transitive.

$$R(1,4) < \max\{\min[R(1,2), R(2,4)], \min[R(1,5), R(5,4)]\} \tag{4.4-4}$$

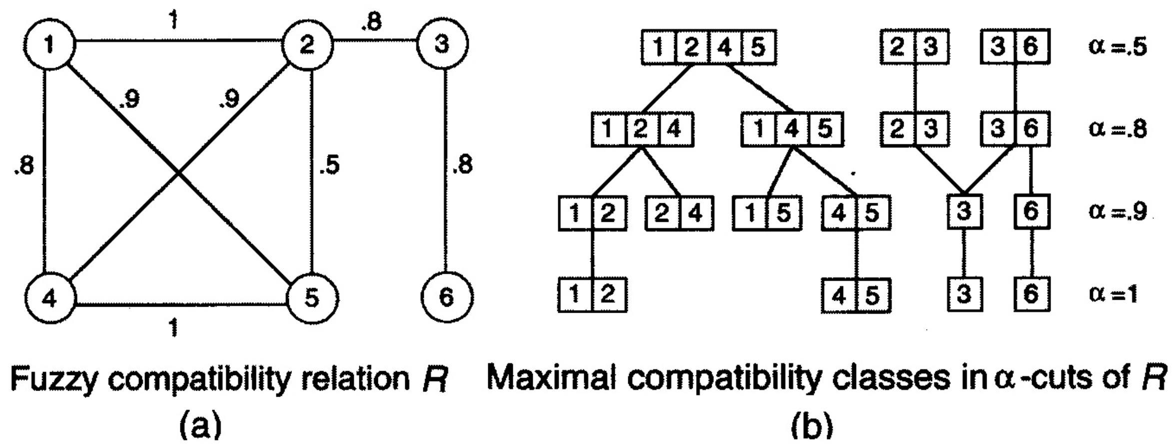

Figure 4.4-2 An example of fuzzy compatibility.

Definition: The **transitive closure** of R is the smallest fuzzy relation that is transitive and contains R.

The transitive closure, R_T, of a fuzzy relation R can be determined by the following two steps:

(1) Compute $R' = R \cup (R \circ R)$;

(2) If $R' \neq R$, rename R' as R and go to step (1); otherwise $R' = R_T$ and the algorithm terminates.

The algorithm guarantees that we obtain the transitive closure in less than *n* - 1 iterations.

Example:

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 0 & 0.8 & 0.9 & 0 \\ 1 & 1 & 0.8 & 0.9 & 0.5 & 0 \\ 0 & 0.8 & 1 & 0 & 0 & 0.8 \\ 0.8 & 0.9 & 0 & 1 & 1 & 0 \\ 0.9 & 0.5 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 1 \end{bmatrix} \tag{4.4-5}$$

$$\mathbf{R} \circ \mathbf{R} = \begin{bmatrix} 1 & 1 & 0.8 & 0.9 & 0.9 & 0 \\ 1 & 1 & 0.8 & 0.9 & 0.9 & 0.8 \\ 0.8 & 0.8 & 1 & 0.8 & 0.5 & 0.8 \\ 0.9 & 0.9 & 0.8 & 1 & 1 & 0 \\ 0.9 & 0.9 & 0.5 & 1 & 1 & 0 \\ 0 & 0.8 & 0.8 & 0 & 0 & 1 \end{bmatrix} \tag{4.4-6}$$

$$\mathbf{R} \cup (\mathbf{R} \circ \mathbf{R}) = \begin{bmatrix} 1 & 1 & 0.8 & 0.9 & 0.9 & 0 \\ 1 & 1 & 0.8 & 0.9 & 0.9 & 0.8 \\ 0.8 & 0.8 & 1 & 0.8 & 0.5 & 0.8 \\ 0.9 & 0.9 & 0.8 & 1 & 1 & 0 \\ 0.9 & 0.9 & 0.5 & 1 & 1 & 0 \\ 0 & 0.8 & 0.8 & 0 & 0 & 1 \end{bmatrix} \tag{4.4-7}$$

$$R \circ R \subseteq R \cup (R \circ R) \tag{4.4-8}$$

Since $R' \neq R$, R' is renamed as R, step (1) is then repeated.

After the second iteration the resulting matrix,

$$\mathbf{R} \cup (\mathbf{R} \circ \mathbf{R}) = \begin{bmatrix} 1 & 1 & 0.8 & 0.9 & 0.9 & 0.8 \\ 1 & 1 & 0.8 & 0.9 & 0.9 & 0.8 \\ 0.8 & 0.8 & 1 & 0.8 & 0.8 & 0.8 \\ 0.9 & 0.9 & 0.8 & 1 & 1 & 0.8 \\ 0.9 & 0.9 & 0.8 & 1 & 1 & 0.8 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 & 1 \end{bmatrix} \tag{4.4-9}$$

Since $R' \neq R$, R' is renamed as R, step (1) is then repeated.

After that, the relation now does not change. This means that the transitive closure is already received and the algorithm terminates.

- The algorithm can be used not only for calculating transitive closures of given fuzzy relations, but also for verifying transitivity of fuzzy relations that are supposed to be transitive. If a given relation is transitive after the algorithm is applied, the algorithm terminates after the first iteration.

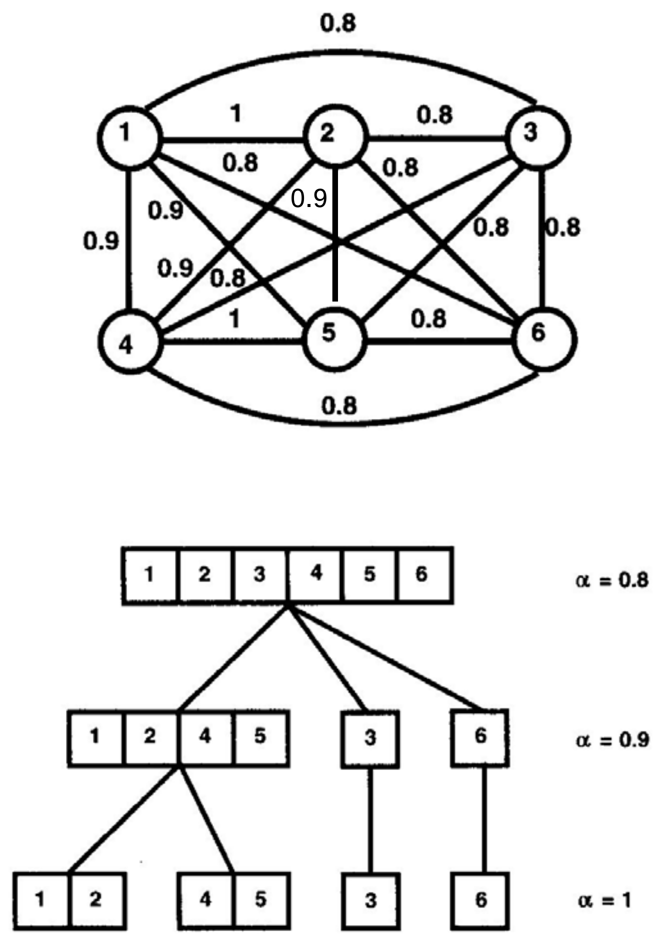

Figure 4.4-3 Equivalence relation that is the transitive closure.

4.5 Fuzzy Partial Orderings

Definition: Fuzzy partial ordering is a fuzzy relation on X that satisfies the reflexivity, transitivity, and antisymmetry.

Definition: A fuzzy relation R on X is ***antisymmetric*** when $R(x, y) > 0$ and $R(y, x) > 0$ imply that $x = y$ for any $x, y \in X$.

Example: $X = \{x_1, x_2, x_3, x_4, x_5\}$

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0.5 & 0 & 0 \\ 0.7 & 1 & 0.9 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0.9 & 1 & 1 & 0.9 \\ 0.7 & 0 & 0.8 & 0 & 1 \end{bmatrix} \tag{4.5-1}$$

- x_1 is preferred to x_3 with the degree of 0.5.
- x_3 is preferred to x_1 with the degree 0.
- x_4 is preferred to x_5 with the degree of 0.9, and so forth.
- In general, x_i is preferred to x_j if $x_i \leq x_j$ according to R.
- The degree of preference of each alternative to itself is defined as 1.
- All α-cuts are crisp partial orderings.
- With increasing values of α, the partial orderings become weaker.
- For $\alpha = 0.1$, the relation is linear, which means that all alternatives are comparable.

- For $\alpha = 0.5$, alternatives 2 and 5 are not comparable.
- The number of pairs that are not comparable increases with increasing values of α.
- For $\alpha = 1$, no pairs are comparable except two: 4 is preferable to both 1 and 3.

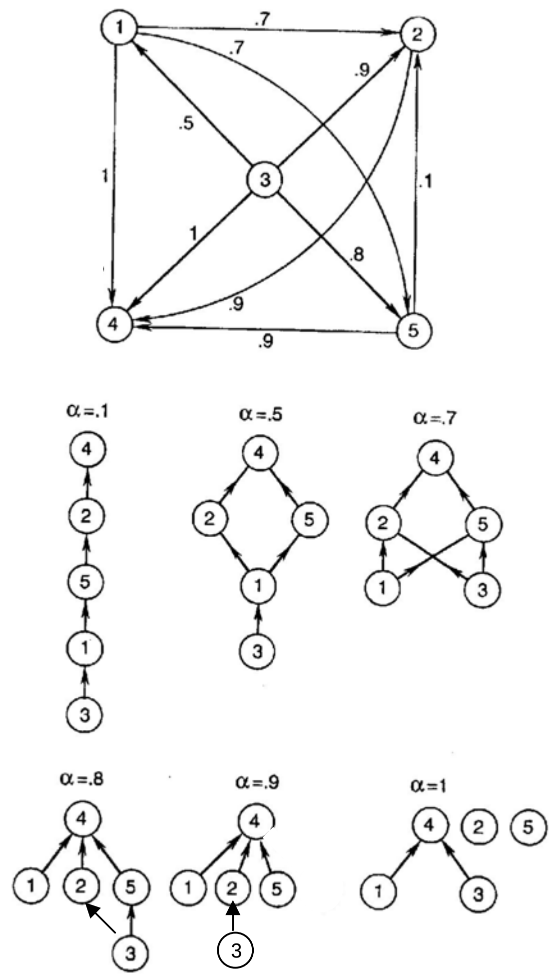

Figure 4.5-1 Example of fuzzy partial ordering.

4.6 Projections and Cylindric Extensions

Definition: For an arbitrary n-dimensional fuzzy relation R on $X = X_1 \times X_2 \times \ldots \times X_n$, the Cartesian product of the chosen dimensions (i.e., some of the sets X_1, X_2, …, X_n) denoted by P and the Cartesian product of the remaining dimensions by $\overline{P}$, and each n-tuple $x \in X$ written in terms of its components $p \in P$ and $\overline{p} \in \overline{P}$,

The **projection** R_p of R **with respect to the dimensions** employed in P for each $p \in P$

$$R_p(p) = \max_{\overline{p} \in \overline{P}} R(p, \overline{p}) \tag{4.6-6}$$

Example: A fuzzy relation Q which expresses the association of a set of symptoms, $S = \{s_1, s_2, s_3\}$ with a set of diseases, $D = \{d_1, d_2, d_3, d_4, d_5\}$.

$$\mathbf{Q} = \begin{bmatrix} 0.7 & 0 & 0 & 0.3 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 & 0 \\ 0 & 0.7 & 0.2 & 0.9 & 0 \end{bmatrix} \tag{4.6-1}$$

The **projection** of the fuzzy relation Q **on the first dimension** (set S), denoted by Q_1, is defined for each $s \in S$.

$$Q_1(s) = \max_{d \in D} Q(s, d) \tag{4.6-2}$$

The **projection** of Q **on the second dimension** (set D), denoted by Q_2, is defined for each $d \in D$.

$$Q_2(d) = \max_{s \in S} Q(s, d) \tag{4.6-3}$$

$$Q_1 = 0.7/s_1 + 0.8/s_2 + 0.9/s_3 \tag{4.6-4}$$

$$Q_2 = 0.7/d_1 + 0.7/d_2 + 0.8/d_3 + 0.9/d_4 + 0.6/d_5 \tag{4.6-5}$$

Definition: The **cylindric extension**, ${}^{EY}R$, of R into Y ,

$${}^{EY}R(x, y) = R(x) \tag{4.6-7}$$

for all $x \in X$ and all $y \in Y$.

Example: Cylindric extensions of the two projections Q_1 and Q_2 of the relation Q with respect to D and S,

$${}^{ED}\mathbf{Q}_1 = \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \\ 0.9 & 0.9 & 0.9 & 0.9 & 0.9 \end{bmatrix} \tag{4.6-8}$$

$${}^{ES}\mathbf{Q}_2 = \begin{bmatrix} 0.7 & 0.7 & 0.8 & 0.9 & 0.6 \\ 0.7 & 0.7 & 0.8 & 0.9 & 0.6 \\ 0.7 & 0.7 & 0.8 & 0.9 & 0.6 \end{bmatrix} \tag{4.6-9}$$

$${}^{ED}\mathbf{Q}_1 \cap {}^{ES}\mathbf{Q}_2 = \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 & 0.6 \\ 0.7 & 0.7 & 0.8 & 0.8 & 0.6 \\ 0.7 & 0.7 & 0.8 & 0.9 & 0.6 \end{bmatrix} \supseteq Q \tag{4.6-10}$$

Fuzzy relation Q is **not a cylindric closure** of the projections.

Example: A binary fuzzy relation defined on $X \times Y$, where $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$,

$$\mathbf{R} = \begin{bmatrix} 0.3 & 0.4 & 0.4 & 0.4 \\ 0.3 & 0.5 & 0.7 & 0.7 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.8 & 0.9 \end{bmatrix} \tag{4.6-11}$$

Projections,

$$R_1 = 0.4/x_1 + 0.7/x_2 + 0.2/x_3 + 0.9/x_4 \tag{4.6-12}$$

$$R_2 = 0.3/y_1 + 0.5/y_2 + 0.8/y_3 + 0.9/y_4 \tag{4.6-13}$$

The cylindric extensions,

$$^{EY}\mathbf{R}_1 = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 \\ 0.7 & 0.7 & 0.7 & 0.7 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.9 & 0.9 & 0.9 & 0.9 \end{bmatrix} \tag{4.6-14}$$

$$^{EX}\mathbf{R}_2 = \begin{bmatrix} 0.3 & 0.5 & 0.8 & 0.9 \\ 0.3 & 0.5 & 0.8 & 0.9 \\ 0.3 & 0.5 & 0.8 & 0.9 \\ 0.3 & 0.5 & 0.8 & 0.9 \end{bmatrix} \tag{4.6-15}$$

The intersection is the original relation R, and, hence, R is **a cylindric closure** of its projections.