# 5. Fuzzy Arithmetic

## 5.1 Fuzzy Numbers

- ***Fuzzy number*** is the number whose value is not absolutely precise numbers.
- A fuzzy number is described in terms of a number word and a linguistic modifier, such as *approximately*, *nearly*, or *around*.

**Examples**: the time is now "*about two o'clock*," a bunch of bananas weighs "*approximately four pounds*."

Every fuzzy number $A$ is expressed by a membership function of the form

$$A : R \rightarrow [0,1] \tag{5.1-1}$$

$$A(x) = \begin{cases} f(x) & \text{for } x \in [a,b] \\ 1 & \text{for } x \in [b,c] \\ g(x) & \text{for } x \in [c,d] \\ 0 & \text{for } x < a \text{ and } x > d \end{cases} \tag{5.1-2}$$

where $a < b < c < d$

$f$: a continuous function that increases to 1 at point $b$

$g$: a continuous function that decreases from 1 at point $c$

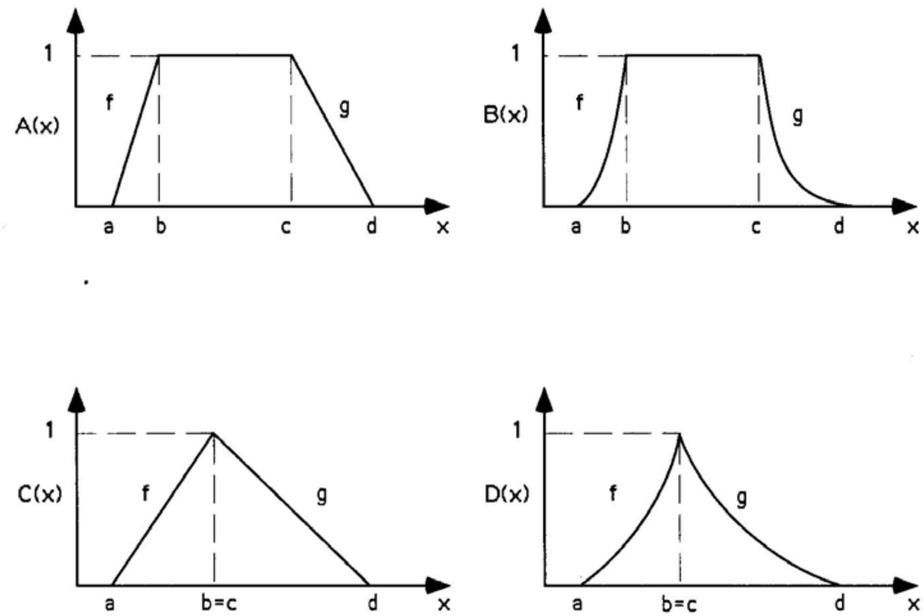



Figure 5.1-1 Example of fuzzy numbers.

- The choice of the real numbers *a, b, c, d* of a fuzzy number is very important and highly dependent on the context of each application.
- Most current applications that employ fuzzy numbers are not significantly affected by the shapes of functions *f* and *g*.
- It is quite natural to choose simple linear functions, represented by straight lines.

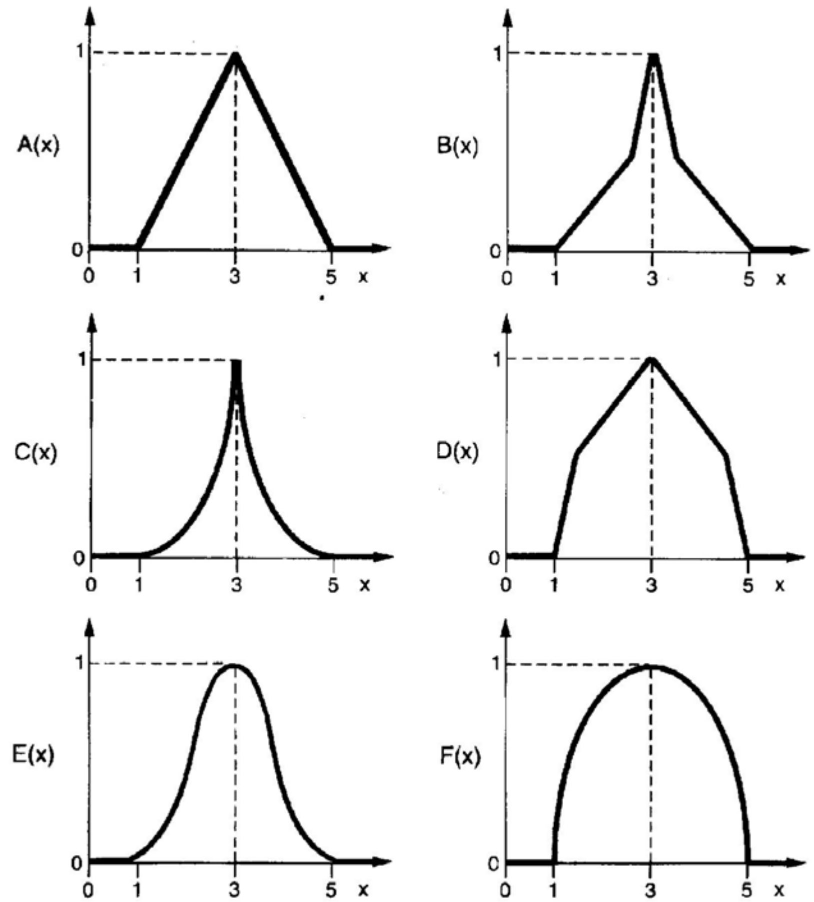



Figure 5.1-2 Possible fuzzy numbers to capture the concept "*around 3*."

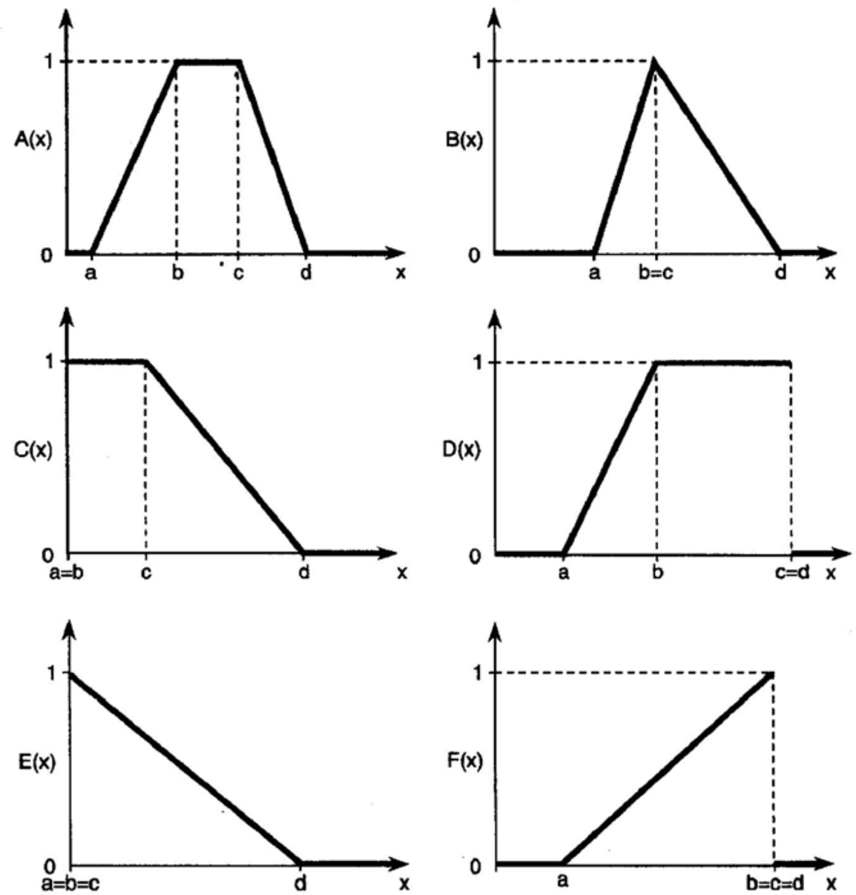



Figure 5.1-3 Trapezoidal-shaped fuzzy number and its various degenerated cases.

**Fuzzy numbers properties**:

1. Fuzzy numbers are normal fuzzy sets (i.e., the core of every fuzzy number is not empty).
2. The $\alpha$-cuts of every fuzzy number are closed intervals of real numbers.
3. The support of every real number is the open interval ($a$, $d$) of real numbers.
4. Fuzzy numbers are convex fuzzy sets.

## 5.2 Arithmetic Operations on Intervals

Consider two closed intervals of real numbers, $[a, b]$ and $[c, d]$, for which $a \le b$ and $c \le d$.

**Addition (+)**

$$[a, b] + [c, d] = [a + c, b + d] \qquad (5.2\text{-}1)$$

**Examples**:

$$[2, 5] + [1, 3] = [2 + 1, 5 + 3] = [3, 8] \qquad (5.2\text{-}2)$$

$$[0, 1] + [-6, 5] = [0 - 6, 1 + 5] = [-6, 6] \qquad (5.2\text{-}3)$$

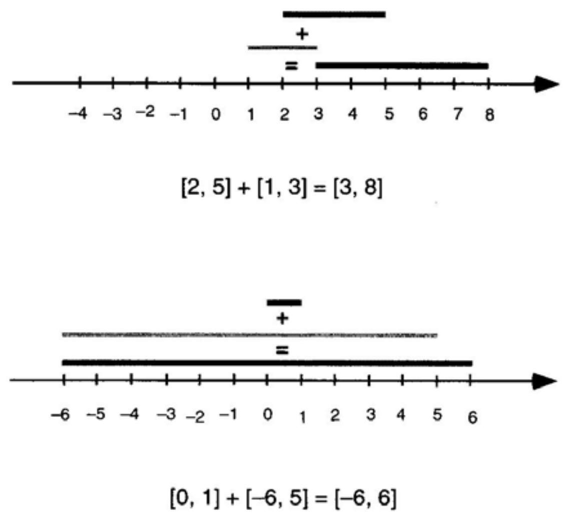



Figure 5.2-1 Interval addition.

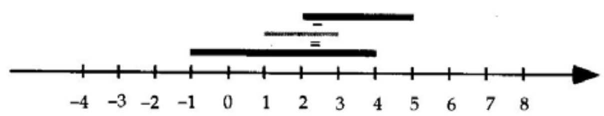
**Subtraction (-)**

$$[a, b] - [c, d] = [a - d, b - c] \quad (5.2\text{-}4)$$

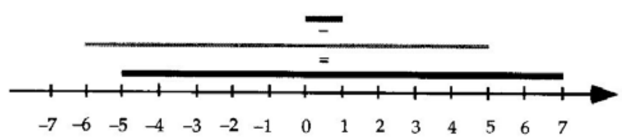
**Examples**:

$$[2, 5] - [1, 3] = [2 - 3, 5 - 1] = [-1, 4] \quad (5.2\text{-}5)$$

$$[0, 1] - [-6, 5] = [0 - 5, 1 + 6] = [-5, 7] \quad (5.2\text{-}6)$$




[2, 5] – [1, 3] = [–1, 4]




[0, 1] – [–6, 5] = [–5, 7]

Figure 5.2-2 Interval subtraction.

**Multiplication (·)**

$$[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)] \quad (5.2\text{-}7)$$

**Examples**:

$$[-1, 1] \cdot [-2, 0.5] = [\min((-1 \cdot -2), (-1 \cdot 0.5), (1 \cdot -2), (1 \cdot 0.5)), \max((-1 \cdot -2), (-1 \cdot 0.5), (1 \cdot -2), (1 \cdot 0.5))]$$

$$= [\min(2, -0.5, -2, 0.5), \max(2, -0.5, -2, 0.5)] = [-2, 2] \quad (5.2\text{-}8)$$

$$[3, 4] \cdot [2, 2] = [\min((3 \cdot 2), (3 \cdot 2), (4 \cdot 2), (4 \cdot 2)), \max((3 \cdot 2), (3 \cdot 2), (4 \cdot 2), (4 \cdot 2))]$$

$$= [\min(6, 6, 8, 8), \max(6, 6, 8, 8)] = [6, 8] \quad (5.2\text{-}9)$$

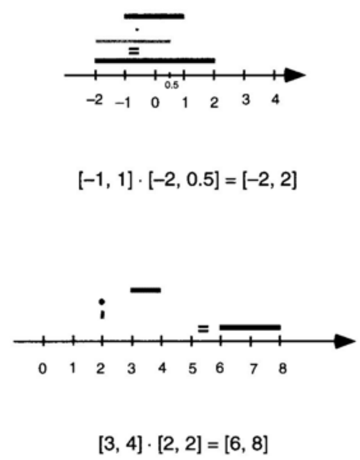



Figure 5.2-3 Interval multiplication.

**Division (/)**

$$[a, b] / [c, d] = [a, b] \cdot [1/d, 1/c] = [\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)] \quad (5.2\text{-}10)$$

The number 0 is not one of the elements in the divisor interval [*c, d*].

**Examples**:

$$[-1, 1] / [-2, -0.5] = [-1, 1] \cdot [1/-0.5, 1/-2] = [\min((-1/-2), (-1/-0.5), (1/-2), (1/-0.5)), \max((-1/-2), (-1/-0.5), (1/-2), (1/-0.5))]$$
$$= [\min(0.5, 2, -0.5, -2), \max(0.5, 2, -0.5, -2)] = [-2, 2] \quad (5.2\text{-}11)$$

$$[4, 10] / [1, 2] = [4, 10] \cdot [1/2, 1/1] = [\min((4/1), (4/2), (10/1), (10/2)), \max((4/1), (4/2), (10/1), (10/2))]$$
$$= [\min(4, 2, 10, 5), \max(4, 2, 10, 5)] = [2, 10] \quad (5.2\text{-}12)$$

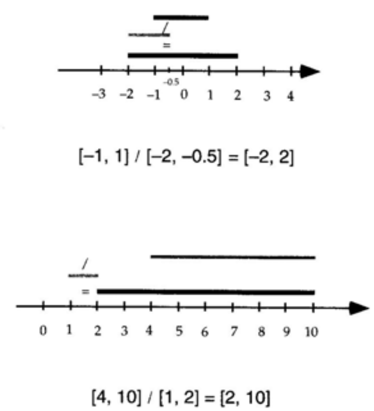



Figure 5.2-4 Interval division.

## 5.3 Arithmetic Operations on Fuzzy Numbers

- Consider the triangular-shape fuzzy numbers *A* and *B* representing *approximately 2* and *approximately 4*,
- In performing addition,
    - The first step is to add the apexes of the two numbers, 2 + 4 = 6.
    - The second step is the sum of the bases, [4, 8] = [1, 3] + [3, 5].
- Subtraction, *A* - *B*, is performed in a similar way.
- The simple procedures are not applicable to multiplication and division.

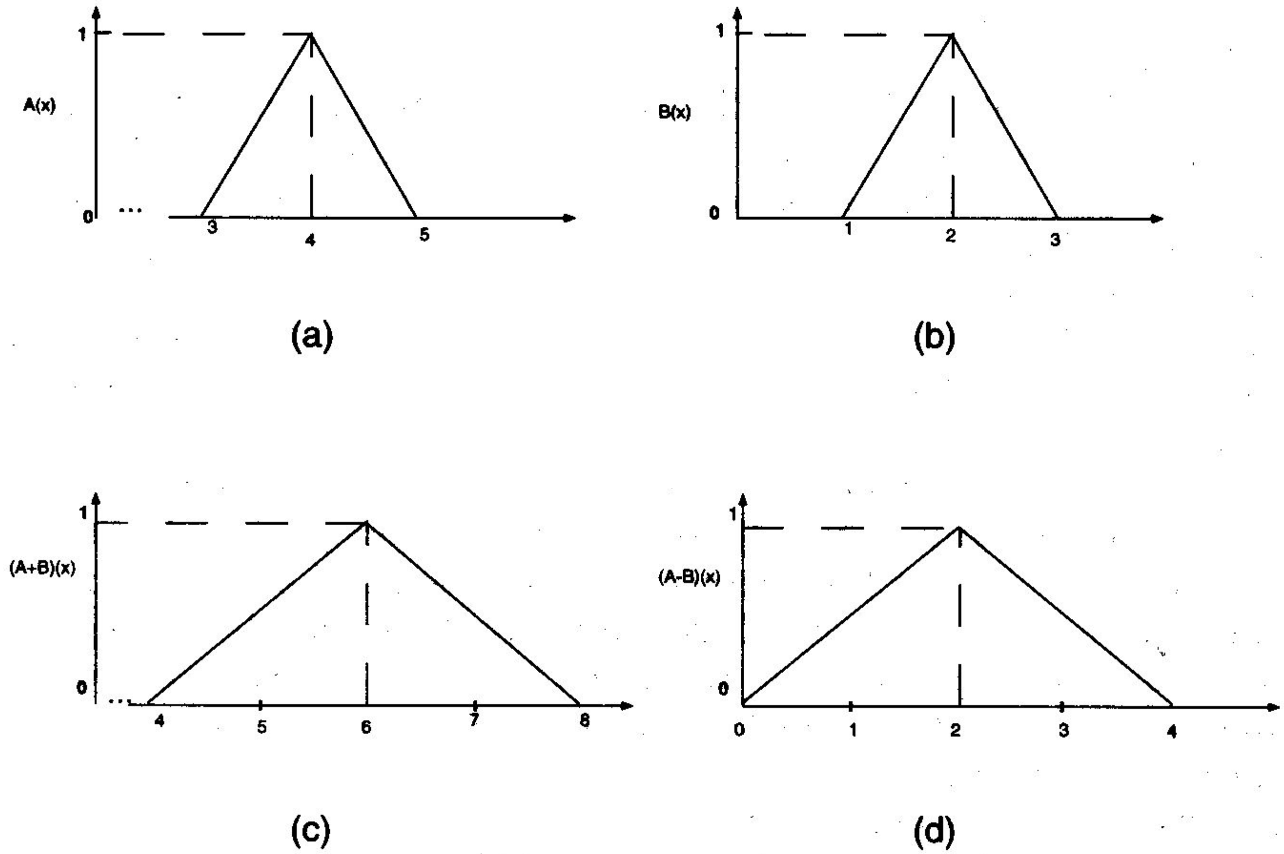



Figure 5.3-1 Simplified addition and subtraction of triangular fuzzy numbers.

Consider arbitrary fuzzy numbers *A* and *B*, and let * denote any of the four interval arithmetic operations.

For each $\alpha \in (0, 1]$, the $\alpha$-cut of $A * B$ is defined in terms of the $\alpha$-cuts of *A* and *B* by

$$^{\alpha}(A * B) = {}^{\alpha}A * {}^{\alpha}B \tag{5.3-1}$$

which is not applicable when * is division and $0 \in {}^{\alpha}B$ for any $\alpha \in (0, 1]$.

The resulting fuzzy number $A * B$,

$$A * B = \bigcup_{\alpha \in [0,1]} {}^{\alpha}(A * B) \cdot \alpha \tag{5.3-2}$$

**Examples**:

$$A(x) = \begin{cases} 0 & \text{for } x < -1 \text{ and } x > 3 \\ (x+1)/2 & \text{for } -1 \le x \le 1 \\ (3-x)/2 & \text{for } 1 \le x \le 3 \end{cases} \tag{5.3-3}$$

$$B(x) = \begin{cases} 0 & \text{for } x < 1 \text{ and } x > 5 \\ (x-1)/2 & \text{for } 1 \le x \le 3 \\ (5-x)/2 & \text{for } 3 \le x \le 5 \end{cases} \tag{5.3-4}$$

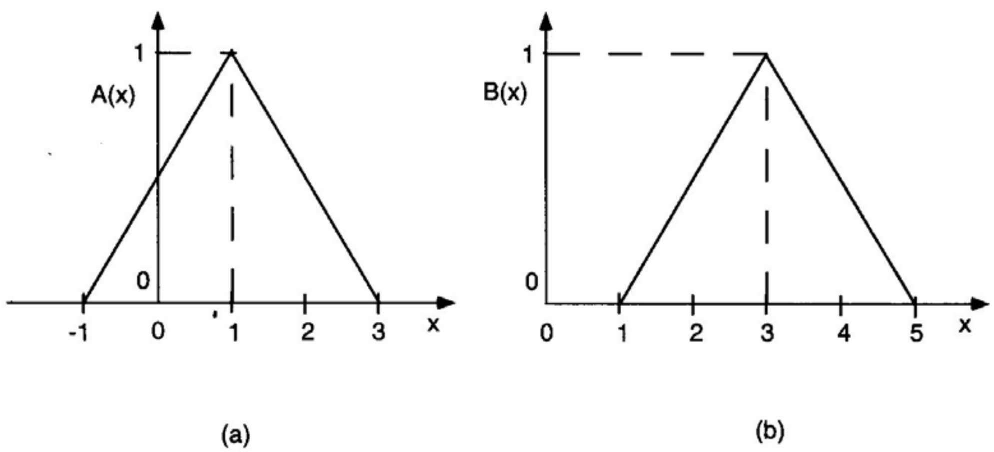



Figure 5.3-2 Fuzzy number used for illustrating arithmetic operations on fuzzy numbers.

For any given $\alpha \in (0, 1]$, the $\alpha$-cuts of $A$ and $B$,

$$^{\alpha}A = [^{\alpha}a_1, {}^{\alpha}a_2] \text{ and } {}^{\alpha}B = [^{\alpha}b_1, {}^{\alpha}b_2] \quad (5.3\text{-}5)$$

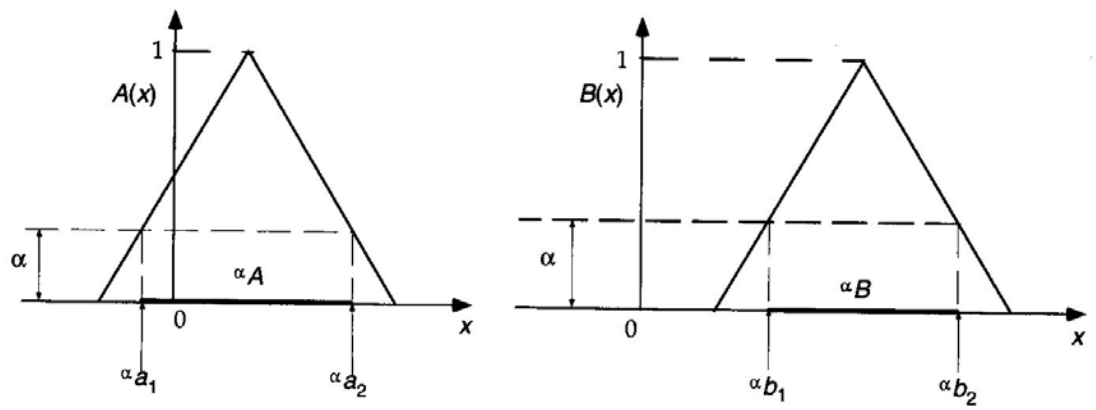



Figure 5.3-3 Construction of α-cuts of fuzzy numbers.

$$A(^{\alpha}a_1) = (^{\alpha}a_1+1)/2 = \alpha \text{ and } A(^{\alpha}a_2) = (3-{}^{\alpha}a_2)/2 = \alpha \quad (5.3\text{-}6)$$

$$^{\alpha}a_1 = 2\alpha-1 \text{ and } {}^{\alpha}a_2 = 3-2\alpha \quad (5.3\text{-}7)$$

$$^{\alpha}A = [2\alpha-1, 3-2\alpha] \quad (5.3\text{-}8)$$

$$B(^{\alpha}b_1) = (^{\alpha}b_1-1)/2 = \alpha \text{ and } B(^{\alpha}b_2) = (5-{}^{\alpha}b_2)/2 = \alpha \quad (5.3\text{-}9)$$

$$^{\alpha}b_1 = 2\alpha+1 \text{ and } {}^{\alpha}b_2 = 5-2\alpha \quad (5.3\text{-}10)$$

$$^{\alpha}B = [2\alpha+1, 5-2\alpha] \quad (5.3\text{-}11)$$

$$^{\alpha}(A * B) = [2\alpha\text{-}1, 3\text{-}2\alpha] * [2\alpha+1, 5\text{-}2\alpha] \quad (5.3\text{-}12)$$

$$^{\alpha}(A + B) = [2\alpha\text{-}1, 3\text{-}2\alpha] + [2\alpha+1, 5\text{-}2\alpha] = [4\alpha, 8\text{-}4\alpha] \quad (5.3\text{-}13)$$

$$4\alpha = x \qquad \text{when } x \in [0, 4] \quad (5.3\text{-}14)$$

$$8\text{-}4\alpha = x \qquad \text{when } x \in [4, 8] \quad (5.3\text{-}15)$$

$$\alpha = x/4 = (A+B)(x) \qquad \text{when } x \in [0, 4] \quad (5.3\text{-}16)$$

$$\alpha = (8\text{-}x)/4 = (A+B)(x) \qquad \text{when } x \in [4, 8] \quad (5.3\text{-}17)$$

$$(A+B)(x) = \begin{cases} 0 & \text{for } x < 0 \text{ and } x > 8 \\ x/4 & \text{for } 0 \le x \le 4 \\ (8-x)/4 & \text{for } 4 \le x \le 8 \end{cases} \quad (5.3\text{-}17)$$

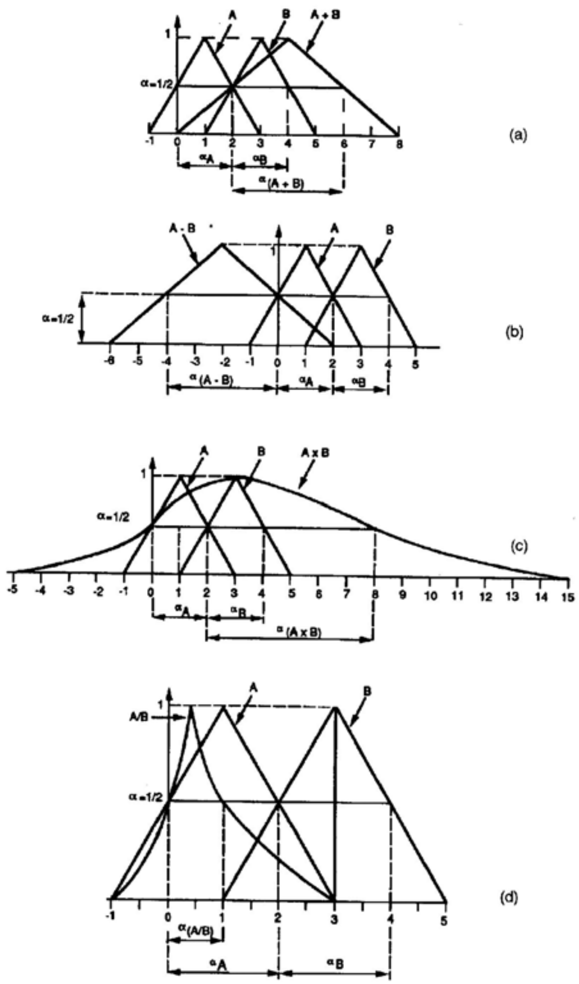



Figure 5.3-4 Arithmetic operations on the fuzzy numbers *A* and *B* specified in figure 5.3-2.

For the difference *A* - *B*,

$$^{\alpha}(A - B) = [2\alpha-1, 3-2\alpha] - [2\alpha+1, 5-2\alpha] = [4\alpha-6, 2-4\alpha] \quad (5.3\text{-}18)$$

$$4\alpha-6 = x \qquad \text{when } x \in [-6, -2] \quad (5.3\text{-}19)$$

$$2-4\alpha = x \qquad \text{when } x \in [-2, 2] \quad (5.3\text{-}20)$$

$$\alpha = (x+6)/4 = (A-B)(x) \qquad \text{when } x \in [-6, -2] \quad (5.3\text{-}21)$$

$$\alpha = (2-x)/4 = (A-B)(x) \qquad \text{when } x \in [-2, 2] \quad (5.3\text{-}22)$$

$$(A-B)(x) = \begin{cases} 0 & \text{for } x < -6 \text{ and } x > 2 \\ (x+6)/4 & \text{for } -6 \le x \le -2 \\ (2-x)/4 & \text{for } -2 \le x \le 2 \end{cases} \quad (5.3\text{-}23)$$

For the product *A* · *B*,

$$^{\alpha}(A \cdot B) = [2\alpha-1, 3-2\alpha] \cdot [2\alpha+1, 5-2\alpha] \quad (5.3\text{-}24)$$

$$^{\alpha}(A \cdot B) = \begin{cases} [-4\alpha^2 + 12\alpha - 5, 4\alpha^2 - 16\alpha + 15] & \text{for } \alpha \in (0, 0.5] \\ [4\alpha^2 - 1, 4\alpha^2 - 16\alpha + 15] & \text{for } \alpha \in (0.5, 1] \end{cases} \quad (5.3\text{-}25)$$

$$-4\alpha^2 + 12\alpha - 5 = x \qquad \text{when } x \in [-5, 0] \quad (5.3\text{-}26)$$

$$4\alpha^2 - 1 = x \qquad \text{when } x \in [0, 3] \quad (5.3\text{-}27)$$

$$4\alpha^2 - 16\alpha + 15 = x \qquad \text{when } x \in [3, 15] \quad (5.3\text{-}28)$$

$$\alpha = \frac{-12 \pm \sqrt{144 - 80 - 16x}}{-8} = \frac{3 - \sqrt{4-x}}{2} \qquad \text{when } x \in [-5, 0] \tag{5.3-29}$$

$$\alpha = \sqrt{\frac{1+x}{4}} = \frac{\sqrt{1+x}}{2} \qquad \text{when } x \in [0, 3] \tag{5.3-30}$$

$$\alpha = \frac{16 \pm \sqrt{256 - 240 + 16x}}{8} = \frac{4 - \sqrt{1+x}}{2} \qquad \text{when } x \in [3, 15] \tag{5.3-31}$$

$$(A \cdot B)(x) = \begin{cases} 0 & \text{for } x < -5 \text{ and } x > 15 \\ [3 - (4-x)^{1/2}]/2 & \text{for } -5 \le x < 0 \\ (1+x)^{1/2}/2 & \text{for } 0 \le x < 3 \\ [4 - (1+x)^{1/2}]/2 & \text{for } 3 \le x \le 15 \end{cases} \tag{5.3-32}$$

For the quotient *A/B*,

$$^{\alpha}(A/B) = \begin{cases} [(2\alpha - 1)/(2\alpha + 1), (3 - 2\alpha)/(2\alpha + 1)] & \text{for } \alpha \in (0, 0.5] \\ [(2\alpha - 1)/(5 - 2\alpha), (3 - 2\alpha)/(2\alpha + 1)] & \text{for } \alpha \in (0.5, 1] \end{cases} \tag{5.3-33}$$

$$\frac{2\alpha - 1}{2\alpha + 1} = x \qquad \text{when } x \in [-1, 0] \tag{5.3-34}$$

$$\frac{2\alpha - 1}{5 - 2\alpha} = x \qquad \text{when } x \in [0, 1/3] \tag{5.3-35}$$

$$\frac{3-2\alpha}{2\alpha+1}=x \qquad \text{when } x \in [1/3,\ 3] \tag{5.3-36}$$

$$2\alpha-1=x(2\alpha+1);\alpha=\frac{1+x}{2-2x} \qquad \text{when } x \in [-1,\ 0] \tag{5.3-37}$$

$$2\alpha-1=x(5-2\alpha);\alpha=\frac{5x+1}{2x+2} \qquad \text{when } x \in [0,\ 1/3] \tag{5.3-38}$$

$$3-2\alpha=x(2\alpha+1);\alpha=\frac{3-x}{2x+2} \qquad \text{when } x \in [1/3,\ 3] \tag{5.3-39}$$

$$(A/B)(x)=\begin{cases} 0 & \text{for } x<-1 \text{ and } x>3 \\ (x+1)/(2-2x) & \text{for } -1\le x<0 \\ (5x+1)/(2x+2) & \text{for } 0\le x<1/3 \\ (3-x)/(2x+2) & \text{for } 1/3\le x\le 3 \end{cases} \tag{5.3-40}$$

# 6. Fuzzy Logic

## 6.1 Introduction

- The term "***fuzzy logic***" is used in two different senses.
    - Fuzzy logic is used as a system of concepts, principles, and methods for dealing with modes of reasoning that are approximate rather than exact.
    - Fuzzy logic is used as a generalization of the various multivalued logic.
- A fuzzy set $A$, the membership degree $A(x)$ for any element $x$ in the underlying universal set $X$ may be interpreted as the degree of truth of the fuzzy proposition "$x$ is a member of $A$."
- The degree of truth of an arbitrary proposition "$x$ is $F$", where $x$ is from the set $X$ and $F$ is a fuzzy linguistic expression may be interpreted as the membership degree $A(x)$ by which a fuzzy set $A$ characterized by the linguistic expression $F$.

## 6.2 Multivalued Logic

### Three-Valued Logic

- Three-valued logic denotes truth, falsity, and indeterminacy by 1, 0, and 1/2, respectively.

For negation (¬),

| $p$ | $\neg p$ |
|---|---|
| 0 | 1 |
| 1/2 | 1/2 |
| 1 | 0 |

Table 6.2-1 Three-valued negation.

For conjunction (∧), disjunction (∨), implication (⇒), and equivalence (⇔),

| | | Łukasiewicz | | | | Bochvar | | | | Kleene | | | | Heyting | | | | Reichenbach | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *a* | *b* | ∧ | ∨ | ⇒ | ⇔ | ∧ | ∨ | ⇒ | ⇔ | ∧ | ∨ | ⇒ | ⇔ | ∧ | ∨ | ⇒ | ⇔ | ∧ | ∨ | ⇒ | ⇔ |
| 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 0 | ½ | 0 | ½ | 1 | ½ | ½ | ½ | ½ | ½ | 0 | ½ | 1 | ½ | 0 | ½ | 1 | 0 | 0 | ½ | 1 | ½ |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| ½ | 0 | 0 | ½ | ½ | ½ | ½ | ½ | ½ | ½ | 0 | ½ | ½ | ½ | 0 | ½ | 0 | 0 | 0 | ½ | ½ | ½ |
| ½ | ½ | ½ | ½ | 1 | 1 | ½ | ½ | ½ | ½ | ½ | ½ | ½ | ½ | ½ | ½ | 1 | 1 | ½ | ½ | 1 | 1 |
| ½ | 1 | ½ | 1 | 1 | ½ | ½ | ½ | ½ | ½ | ½ | 1 | 1 | ½ | ½ | 1 | 1 | ½ | ½ | 1 | 1 | ½ |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | ½ | ½ | 1 | ½ | ½ | ½ | ½ | ½ | ½ | ½ | 1 | ½ | ½ | ½ | 1 | ½ | ½ | ½ | 1 | ½ | ½ |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 6.2-2 Connectives of some three-valued logics.

None of the three-valued logics satisfies the ***law of contradiction*** ($p \wedge \neg p = 0$), the ***law of the excluded middle*** ($p \vee \neg p = 1$), and some other tautologies.

- A logic formula which does not assume the truth value 0 (FALSE) is a ***quasi-tautology***.
- For three-valued logic, quasi-tautology has the truth value of 0.5 or 1.
- A logic which does not assume the truth value 1 (TRUE) is a ***quasi-contradiction***.
- For three-valued logic, quasi-contradiction has the truth value of 0.5 or 0.

For the De Morgan's laws,

| p | q | ¬ | (p | ∧ | q) | ⇔ | (¬p | ∨ | ¬q) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1/2 | 1/2 | 0 | 1/2 | 1/2 | 1/2 | 1 | 1/2 | 1/2 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| 1/2 | 0 | 1/2 | 1/2 | 1/2 | 0 | 1/2 | 1/2 | 1/2 | 1 |
| 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 |
| 1/2 | 1 | 1/2 | 1/2 | 1/2 | 1 | 1/2 | 1/2 | 1/2 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1/2 | 1/2 | 1 | 1/2 | 1/2 | 1/2 | 0 | 1/2 | 1/2 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |

Table 6.2-3 Bochvar Three-valued interpretation of De Morgan's law

| p | q | ¬ | (p | ∧ | q) | ⇔ | (¬p | ∨ | ¬q) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1/2 | 1 | 0 | 0 | 1/2 | 1 | 1 | 1 | 1/2 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| 1/2 | 0 | 1 | 1/2 | 0 | 0 | 1 | 1/2 | 1 | 1 |
| 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1 | 1/2 | 1/2 | 1/2 |
| 1/2 | 1 | 1/2 | 1/2 | 1/2 | 1 | 1 | 1/2 | 1/2 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1/2 | 1/2 | 1 | 1/2 | 1/2 | 1 | 0 | 1/2 | 1/2 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |

Table 6.2-4 Lukasiewicz Three-valued interpretation of De Morgan's law

| p | q | ¬ | (p | ∧ | q) | ⇔ | (¬p | ∨ | ¬q) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1/2 | 1 | 0 | 0 | 1/2 | 1 | 1 | 1 | 1/2 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| 1/2 | 0 | 1 | 1/2 | 0 | 0 | 1 | 1/2 | 1 | 1 |
| 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 |
| 1/2 | 1 | 1/2 | 1/2 | 1/2 | 1 | 1/2 | 1/2 | 1/2 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1/2 | 1/2 | 1 | 1/2 | 1/2 | 1/2 | 0 | 1/2 | 1/2 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |

Table 6.2-5 Kleene Three-valued interpretation of De Morgan's law

For Modus ponens,

$$[(p \Rightarrow q) \wedge p] \Rightarrow q \qquad (6.2\text{-}1)$$

| $p$ | $q$ | $[(p$ | $\Rightarrow$ | $q)$ | $\wedge$ | $p]$ | $\Rightarrow$ | $q$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 1/2 | 0 | 1 | 1/2 | 0 | 0 | 1 | 1/2 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1/2 | 0 | 1/2 | 1/2 | 0 | 1/2 | 1/2 | 1/2 | 0 |
| 1/2 | 1/2 | 1/2 | 1 | 1/2 | 1/2 | 1/2 | 1 | 1/2 |
| 1/2 | 1 | 1/2 | 1 | 1 | 1/2 | 1/2 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1/2 | 1 | 1/2 | 1/2 | 1/2 | 1 | 1 | 1/2 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 6.2-6 Lukasiewicz interpretation of modus ponens

## *n*-Valued Logic

- **Many-valued logic** is usually referred to as ***n-valued logic***, where $n$ is the number of truth values.
- A proposition $p$ may have a truth value of 1/2 for halfaway between completely true and completely false, or a truth value of 3/4 for mostly true and only a little false, or a truth value of 1/4 for mostly false, or a truth value of 7/8 for hardly false.
- The set of truth values of an $n$-valued logic, $T_n$,

$$T_n = \left\{ \frac{0}{n-1}, \frac{1}{n-1}, \cdots, 1 \right\} \tag{6.2-2}$$

- In a five-valued logic, the truth values are 0, 1/4, 1/2, 3/4, and 1.

For $n$-valued logic proposed by the Polish logician Lukasiewicz for any $n \geq 2$ (i.e., $n$ may go to infinity),

$$\bar{p} = 1 - p \tag{6.2-3}$$

$$p \wedge q = \min(p, q) \tag{6.2-4}$$

$$p \vee q = \max(p, q) \tag{6.2-5}$$

$$p \Rightarrow q = \min(1, 1 - p + q) \tag{6.2-6}$$

$$p \Leftrightarrow q = 1 - |p - q| \tag{6.2-7}$$

- When the truth values include all real numbers in the unit interval [0, 1], the logic is called an ***infinite-valued logic*** or a ***continuous logic***.

- Reasoning with propositions involving imprecise concepts is usually referred to as ***approximate reasoning***.

**Example of approximate reasoning**:

*Old coins are usually rare collectibles.*

*Rare collectibles are expensive.*

______________________________ (6.2-8)

$\therefore$ *Old coins are usually expensive.*

The linguistic expressions may contain fuzzy linguistic terms of several types, including

- ***fuzzy predicates***, such as tall, young, small, medium, normal, expensive, near, intelligent, and the like
- ***fuzzy truth values***, such as true, false, fairly true, or very true
- ***fuzzy probabilities***, such as likely, unlikely, very likely, or highly unlikely
- ***fuzzy quantifiers***, such as many, few, most, or almost all

## 6.3 Fuzzy Propositions

- unconditional and unqualified propositions
- unconditional and qualified propositions
- conditional and unqualified propositions
- conditional and qualified propositions

**Unconditional and Unqualified Propositions**

$$p\text{: } X \text{ is } A \tag{6.3-1}$$

$$p\text{: ‘}X \text{ is } A\text{’ is true} \tag{6.3-2}$$

- The membership grade is interpreted as the degree of truth, $T(p_x)$, of proposition

$$p\text{: } X = x \text{ is } A \tag{6.3-3}$$

$$T(p_x) = A(x) \tag{6.3-4}$$

**Example**: $X$: the relative humidity (measured in %), *high humidity*: expressed by membership function $H$,

$$p\text{: } X \text{ is } H \tag{6.3-5}$$

For each $x \in [0, 100]$, the degree of truth, $T(p_x)$, of the fuzzy proposition

$$p_x\text{: } X = x \text{ is } H \tag{6.3-6}$$

is defined by the identity function $T$ from [0, 1] to [0, 1].

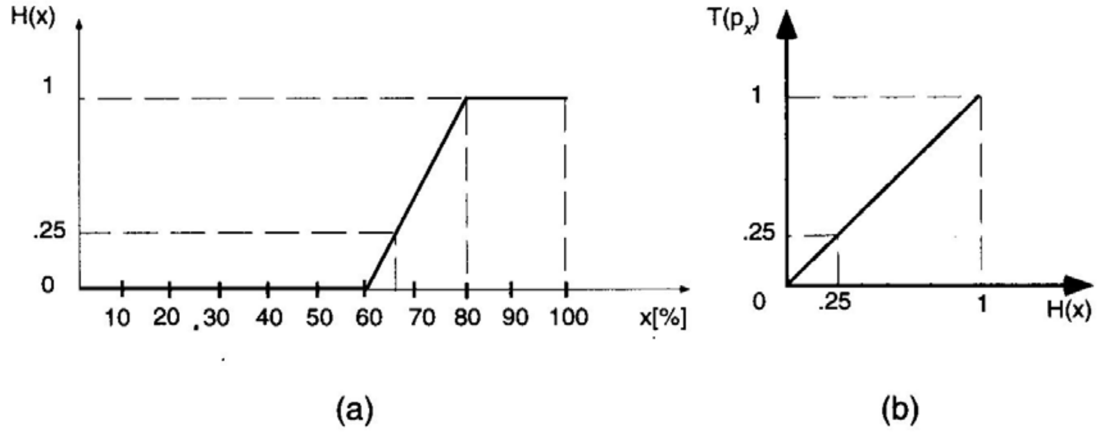



Figure 6.3-1 The degree of truth of the proposition “Humidity is high” when the measured humidity is 65%

If $x = 65$, then $H(65) = 0.25$,

$$T(p_{65}) = H(65) = 0.25 \tag{6.3-7}$$

$$P_{65}\text{: Humidity of 65\% is high} \tag{6.3-8}$$

is true to the degree of 0.25.

## Unconditional and Qualified Propositions

$$p\text{: ‘}X \text{ is } A\text{’ is } S \quad (6.3\text{-}9)$$

*S*: a fuzzy truth qualifier, a linguistic expression that adds a modifier to the claim of simple truth.

- Examples of truth qualifiers are linguistic expressions such as *very true, fairly true*, *false*, *very false*, or *fairly false*.
- Each truth qualifier is characterized by a function from [0, 1] to [0, 1].

The degree of truth, $T_s(p_x)$, of the truth-qualified proposition

$$p_x\text{: ‘}X = x \text{ is } A\text{’ is } S \quad (6.3\text{-}10)$$

$$T_s(p_x) = S(A(x)) \quad (6.3\text{-}11)$$

- True: S(a) = a Very True: $S(a) = a^2$
- Fairly True: $S(a) = a^{1/2}$
- False: S(a) = 1-a
- Very False: $S(a) = (1-a)^2$
- Fairly False: $S(a) = (1-a)^{1/2}$

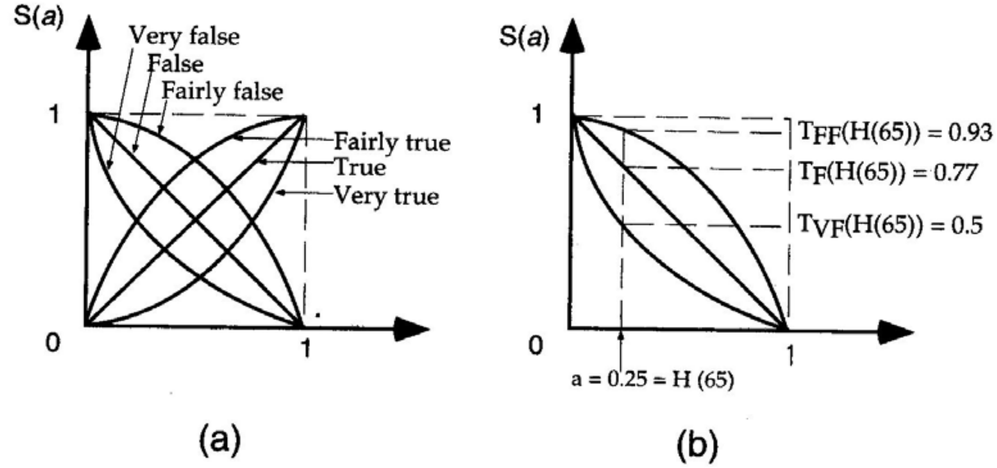



Figure 6.3-2 Illustration of the role of truth qualification.

**Example**: If the actual value of humidity is 65%. $H(65) = 0.25$,

$$P_{65}\text{: 'Humidity of 65\% is high' is } S \quad (6.3\text{-}12)$$

$$T_s(p_{65}) = S(0.25) \quad (6.3\text{-}13)$$

## Conditional and Unqualified Propositions

$$p\text{: If } X \text{ is } A\text{, then } Y \text{ is } B \tag{6.3-14}$$

$$p\text{: 'If } X \text{ is } A\text{, then } Y \text{ is } B\text{' is true} \tag{6.3-15}$$

$$P_{x,y}\text{: 'If } A(x)\text{, then } B(y)\text{' is true} \tag{6.3-16}$$

$$A(x) \Rightarrow B(y) \tag{6.3-17}$$

For each $x \in X$ and each $y \in Y$, the Lukasiewicz implication $I$ determines the degree of truth of the conditional proposition

$$T(p_{x,y}) = I[A(x), B(y)] = \min[1, 1-A(x)+B(y)] \tag{6.3-18}$$

**Example**:

$$p\text{: If a textbook is large, then it is expensive.} \tag{6.3-19}$$

$$T(p_{x,y}) = \min[1, 1-L(x)+E(y)] \tag{6.3-20}$$

$$T(p_{600,45}) = \min[1, 1-1+0.5] = 0.5 \tag{6.3-21}$$

$$T(p_{450,42}) = \min[1, 1-0.75+0.4] = 0.65 \tag{6.3-22}$$

$$T(p_{574,60}) = \min[1, 1-1+1] = 1 \tag{6.3-23}$$

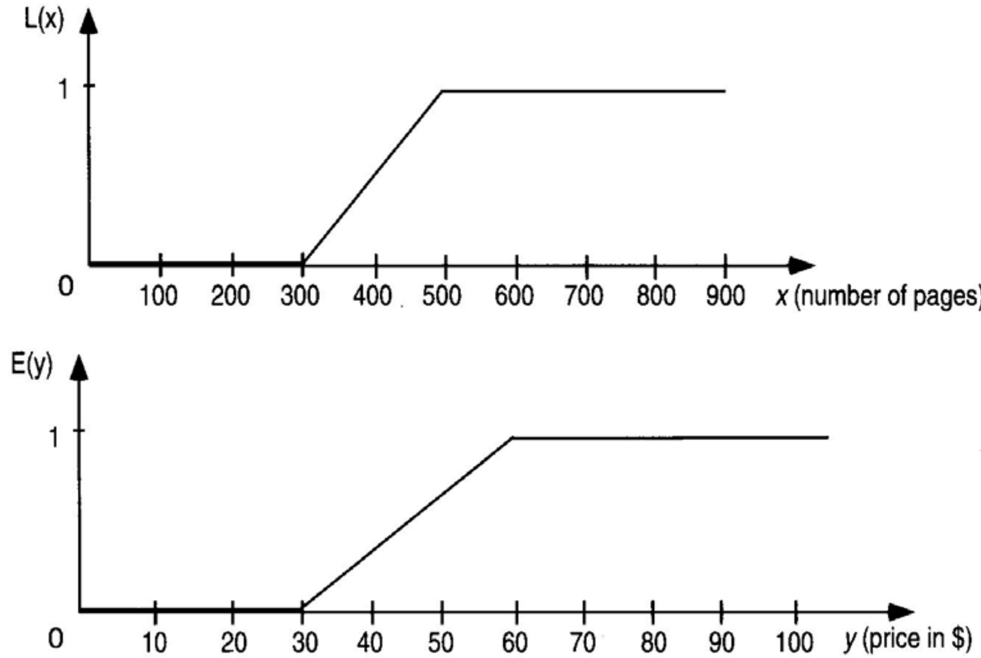



Figure 6.3-3 Fuzzy sets involved in conditional fuzzy proposition of the proposition

"If a textbook is large, then it is expensive."

**Conditional and Qualified Propositions**

$$p\text{: 'If } X \text{ is } A\text{, then } Y \text{ is } B\text{' is } S \tag{6.3-24}$$

$S$: a truth qualifier.

$$T_s(p_{x,y}) = S[T(p_{x,y})] \tag{6.3-25}$$

**Example**:

$$p\text{: 'If a textbook is large, then it is expensive' is very true} \tag{6.3-26}$$

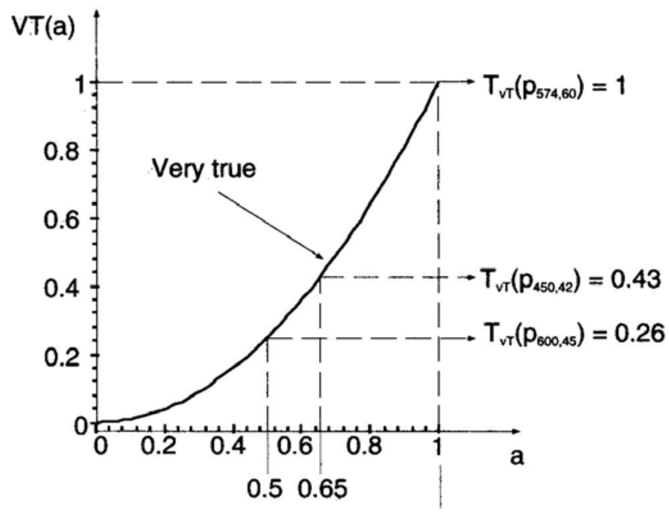



Figure 6.3-4 Degrees of truth of conditional and truth-qualified fuzzy propositions.

## 6.4 Approximate Reasoning

The rule of modus ponens,

$$\begin{array}{l} p \Rightarrow q \\ p \\ \hline q \end{array} \tag{6.4-1}$$

$$[(p \Rightarrow q) \wedge p] \Rightarrow q \tag{6.4-2}$$

**Example of approximate reasoning**:

| | |
|---|---|
| Rule: | If a book is large, then it is expensive |
| Fact: | Book *x* is fairly large |
| Conclusion: | Book *x* is fairly expensive |

(6.4-3)

| | |
|---|---|
| Rule: | If *x* is *A*, then *y* is *B* |
| Fact: | *x* is *A'* |
| Conclusion: | *y* is *B'* |

(6.4-4)

*x*, *y*: variables taking values in the universal sets *X* and *Y*

*A* and *A'*: fuzzy sets on *X*

*B* and *B'*: fuzzy sets on *Y*

Conclusion *B'* is calculated for any $y \in Y$,

$$B'(y) = \sup_{x \in X} \min(A'(x), I(A(x), B(y))) \tag{6.4-5}$$

- This rule is referred to as the ***compositional rule of inference***, where *I* denotes an appropriate fuzzy implication.

# 7. Application: A Survey

## 7.1 Applications

- Control
- Decision Making
- Knowledge Retrieving System

**Fuzzy Controller**

- No mathematic model required, Applicable for both linear and nonlinear system.
- ***In consumer products***: washing machines, vacuum cleaners, electric shavers, dishwashers, rice cookers, video camcorders, cars (for anti-skid brakes, automatic transmissions, speed controls, and other functions), refrigerators, humidifiers, and air conditioners.
- Other applications: for controlling groups of elevators, trains of subway systems, traffic in cities, and various industrial processes, controlling a helicopter according to instructions in natural language which are communicated to the helicopter from ground via wireless transmitter.
- Fuzzy controllers may also be combined with classical controllers to achieve a high performance.
- Fuzzy controllers can be combined with appropriate neural networks whose learning capabilities are utilized for making the controllers adaptive to varying conditions.

## 7.2 Illustrative Examples

### *Fuzzy washing machine*

- Used to determine the proper operating time of the washing machine for each load of clothes.
- The operating time of a washing machine depends on two properties of each given load of clothes.
  - o How dirty the clothes: The degree of dirtiness is measured by the degree of water transparency
  - o The type of soil: A saturation time is used to determine the type of soil.

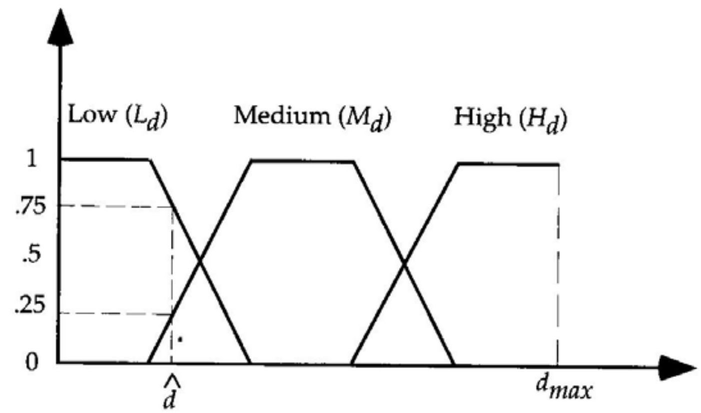



Figure 7.2-1 Fuzzy numbers representing three levels of dirtiness: *low* ($L_d$), *medium* ($M_d$), and *high* ($H_d$). These are states of linguistic variable *D*.

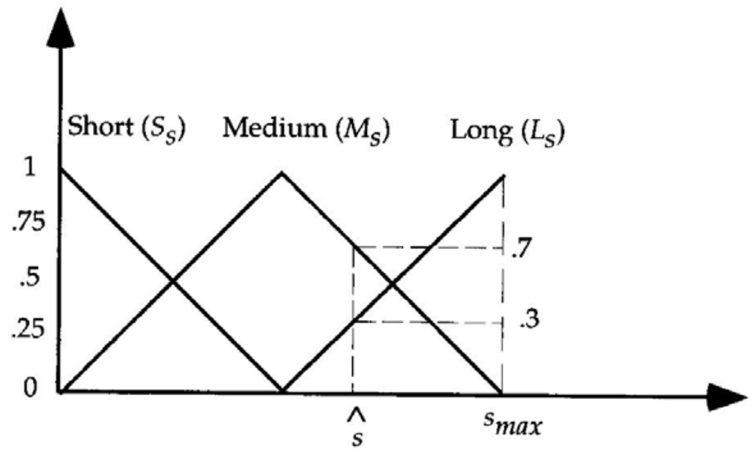



Figure 7.2-2 Fuzzy numbers representing *short* ($S_s$), *medium* ($M_s$), and *long* ($L_s$) saturation time. These are states of linguistic variable *S*.

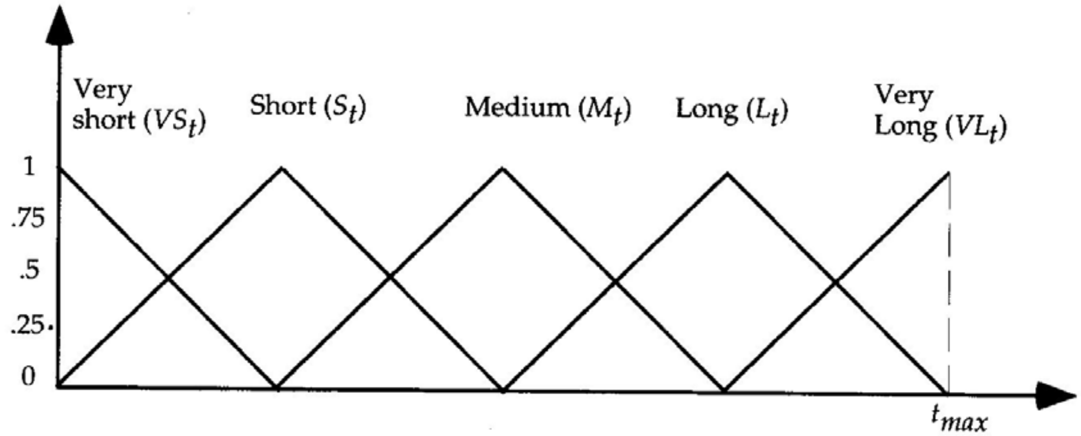



Figure 7.2-3 Fuzzy numbers characterizing the required washing time. These are states of linguistic variable *T*.

- The washing time is determined based on knowledge and experience of the designer expressed by conditional fuzzy propositions of the form.

If $D = x$ and $S = y$, then $T = z$

- In each rule, the states of *D* and *S* are called antecedents and the state of *T* is called the consequent.
- The propositions are usually called ***fuzzy inference rules*** or ***fuzzy if-then rules***.

**Examples of three of the fuzzy inference rules**:

If $D = L_d$ and $S = S_s$, then $T = VS_t$

If $D = M_d$ and $S = M_s$, then $T = M_t$

If $D = H_d$ and $S = L_s$, then $T = VL_t$

| | | S | | |
|---|---|---|---|---|
| | $t$ | $S_s$ | $M_s$ | $L_s$ |
| $\mathcal{D}$ | $L_d$ | $VS_t$ | $S_t$ | $M_t$ |
| | $M_d$ | $S_t$ | $M_t$ | $L_t$ |
| | $H_d$ | $M_t$ | $L_t$ | $VL_t$ |

Figure 7.2-4 Inference rules for fuzzy washing machine.

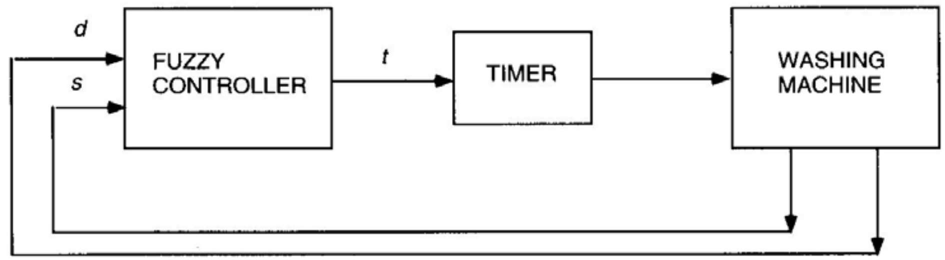



Figure 7.2-5 Fuzzy control of washing time.

For given values of variables *d* and *s*, the controller determines the proper value of variable *t* (washing time) by executing the following steps.

**Step 1**. When specific measured values of the input variables *d* and *s*, denoted by $\hat{d}$ and $\hat{s}$, are received by the controller at some predefined time, their compatibilities with the corresponding antecedents of all inference rules are determined. The rules for which the compatibilities of the measured values with both antecedents are positive are usually referred to as rules that **fire**.

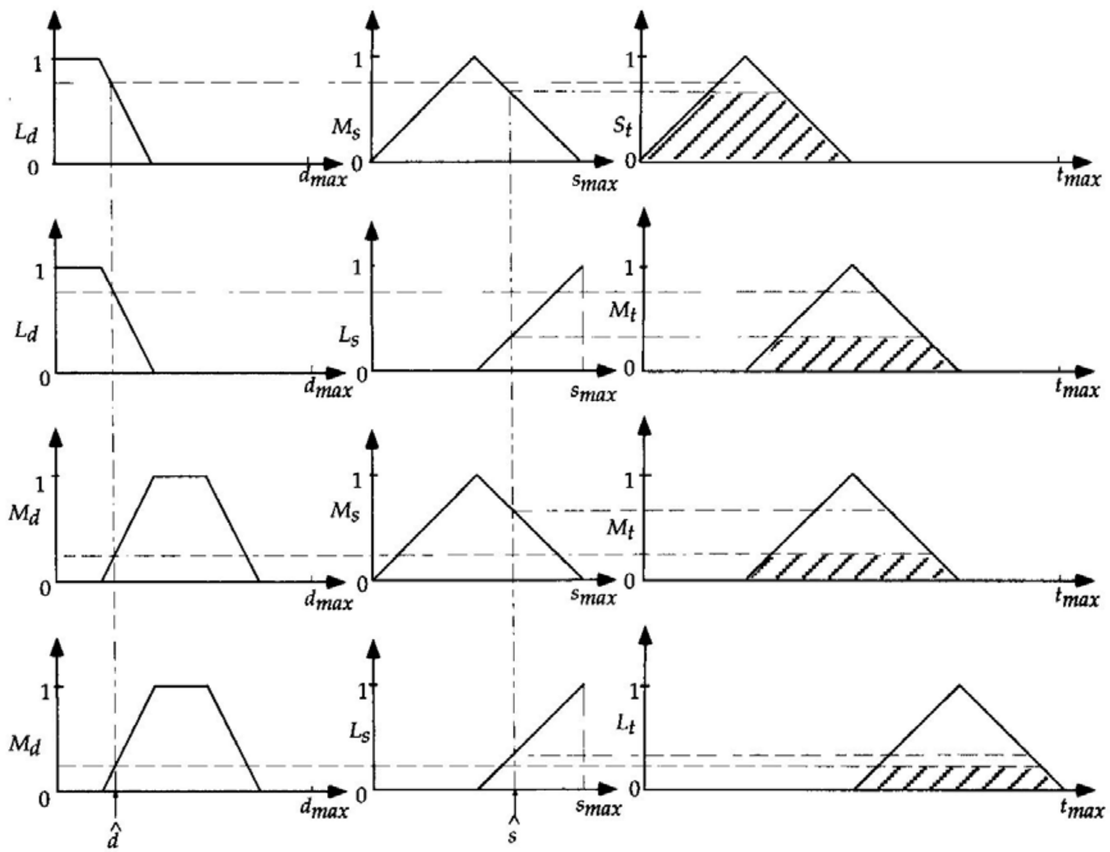



Figure 7.2-6 Inference rules that fire for $d = \hat{d}$ and $s = \hat{s}$.

**Step 2**. An inference is made by each rule that fires.

$$t = f(d, s) \tag{7.2-1}$$

The intersection of the cylindric extensions of the truncated antecedents is then truncated by the minimum degree of compatibility (assuming the standard fuzzy intersection) and this truncation is inherited in the consequent of the rule by the extension principle.

**Step 3**. The overall conclusion is obtained by taking the union of all the individual conclusions.

The overall conclusion is the fuzzy set $C_{\hat{d},\hat{s}}$ whose membership function is defined for each $x \in [0, t_{max}]$ by the formula

$$C_{\hat{d},\hat{s}}(t) = \max\{\min[L_d(\hat{d}), M_s(\hat{s}), VS_t(t)], \min[L_d(\hat{d}), L_s(\hat{s}), S_t(t)],$$

$$\min[M_d(\hat{d}), M_s(\hat{s}), M_t(t)], \min[M_d(\hat{d}), L_s(\hat{s}), L_t(t)]\} \tag{7.2-2}$$

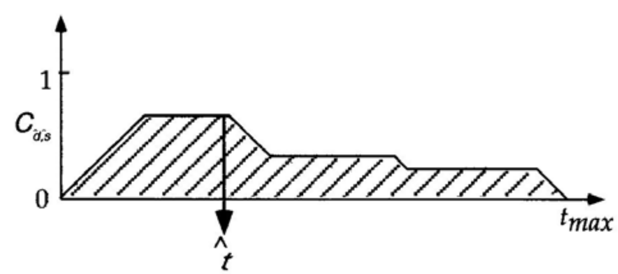



Figure 7.2-7 The fuzzy set which represents the overall conclusion for the measured values $\hat{d}$ and $\hat{s}$ and its defuzzified value $\hat{t}$.

**Step 4**. This last step is called ***defuzzification***. Its purpose is to convert the fuzzy set representing the overall conclusion into a real number that, in some sense, best represents the fuzzy set. The most common method is to determine the value for which the area under the graph of the membership function is equally divided. This method is called a ***center of gravity defuzzification method***.

A fuzzy set $A$ defined on the interval $[a_1, a_2]$,

The center-of-gravity defuzzification, $a$, of $A$

$$a = \frac{\int_{a_1}^{a_2} xA(x)dx}{\int_{a_1}^{a_2} A(x)dx} \tag{7.2-3}$$

$$\hat{t} = \frac{\int_{a_1}^{a_2} xC_{\hat{d},\hat{s}}(t)dt}{\int_{a_1}^{a_2} C_{\hat{d},\hat{s}}(t)dt} \tag{7.2-4}$$

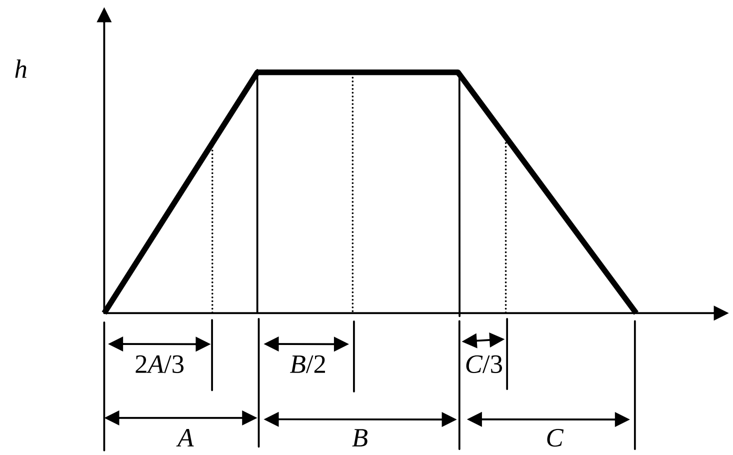
For the triangular or trapezoidal membership functions,

$$a = \frac{\sum_{i=1}^{n} \bar{x}_i A_i}{\sum_{i=1}^{n} A_i} \qquad (7.2.5)$$

$\bar{x}_i$ : center of gravity of area number $i$

$A_i$: area $i$

**Example**:




$$a = \frac{\left(\frac{2}{3}A\right)\left(\frac{1}{2}Ah\right) + \left(A + \frac{1}{2}B\right)(Bh) + \left(A + B + \frac{1}{3}C\right)\left(\frac{1}{2}Ch\right)}{\left(\frac{1}{2}Ah\right) + (Bh) + \left(\frac{1}{2}Ch\right)} \qquad (7.2\text{-}6)$$

## Decision Making

**Example**: Buying a House

- There are only five houses available.
- Criteria of selection of an attractive house:
  - Price
  - Location (distance to work)
  - Real estate taxes
  - Attractiveness
  - Quality of the associated school system

| House | Price (in thousand of dollars) | Distance to Work (in miles) | Taxes (in hundreds of dollars) |
|---|---|---|---|
| *a* | 120 | 30 | 15 |
| *b* | 70 | 5 | 25 |
| *c* | 80 | 10 | 25 |
| *d* | 90 | 25 | 20 |
| *e* | 100 | 20 | 20 |

Table 7.2-1 Basic Information about five houses.

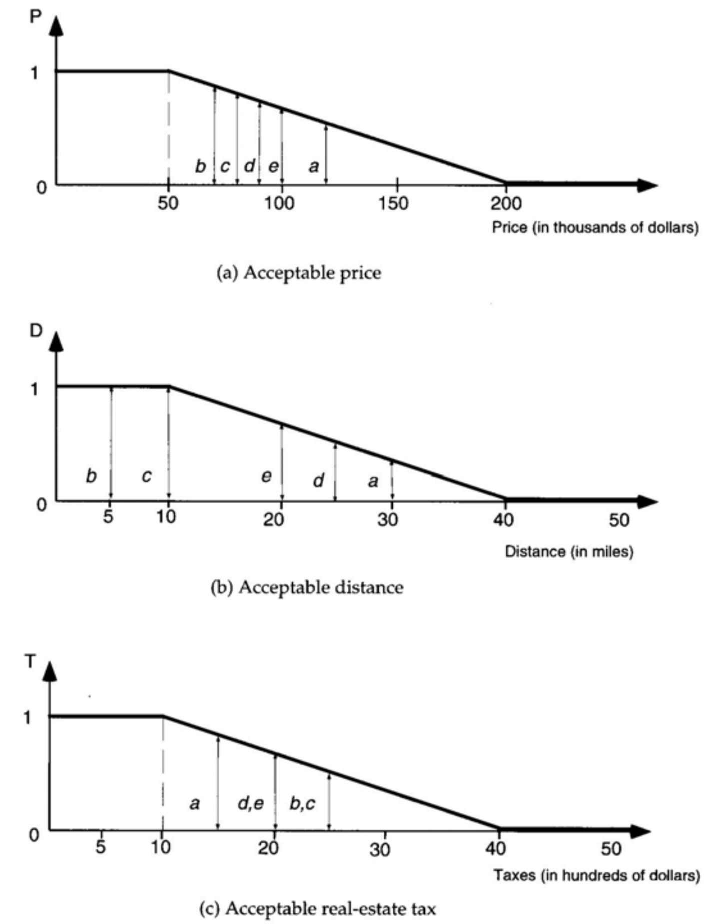



Figure 7.2-8 Fuzzy sets defined by the buyer of a house.

| House | Degree of Acceptability of the Houses According to the Individual Criteria | | | | | Degree of Overall Acceptability |
|---|---|---|---|---|---|---|
| $x$ | $P(x)$ | $D(x)$ | $T(x)$ | $A(x)$ | $S(x)$ | |
| $a$ | 0.53 | 0.33 | 0.83 | 1 | 0.5 | 0.33 |
| $b$ | 0.87 | 1 | 0.5 | 0.5 | 1 | 0.5 |
| $c$ | 0.8 | 1 | 0.5 | 0.6 | 1 | 0.5 |
| $d$ | 0.73 | 0.5 | 0.67 | 0.2 | 0.9 | 0.2 |
| $e$ | 0.67 | 0.67 | 0.67 | 0.8 | 0.9 | 0.67 |

Table 7.2-2 Decision-making example: a summary.

## Knowledge Retrieving System

**Example**: Finding the related documents with the keywords

- There are 15 documents.

$$D = \{d_1, d_2, ..., d_{15}\} \tag{7.2-7}$$

- There are 7 index terms.

  $t_1$-fuzzy number

  $t_2$-interval arithmetic

  $t_3$-fuzzy arithmetic

  $t_4$-convexity

  $t_5$-fuzzy relation

  $t_6$-fuzzy set

  $t_7$-fuzzy proposition

$$T = \{t_1, t_2, ..., t_7\} \tag{7.2-8}$$

The relevance of the index terms to the individual documents in set $D$ is expressed by the matrix of a fuzzy relevance relation $R$ on $T \times D$,

$$
\mathbf{R} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \end{array}
\begin{array}{|ccccccccccccccc|}
d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 & d_9 & d_{10} & d_{11} & d_{12} & d_{13} & d_{14} & d_{15} \\
0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0.3 & 0 & 0 & 0.9 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 1 \\
1 & 0.7 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.9 & 0.5 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0.5 & 0 & 0 & 0 & 0 & 0.4 & 0.8 & 0.4 & 0.2 & 1 & 1 & 0
\end{array} \tag{7.2-9}
$$

The closeness in meanings of the index terms is expressed by the matrix of a fuzzy relation $M$ on $T \times T$,

$$
\mathbf{M} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \end{array}
\begin{array}{|ccccccc|}
t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 \\
1 & 0.5 & 1 & 0.3 & 0 & 0.8 & 0 \\
0.5 & 1 & 0.7 & 0.1 & 0 & 0 & 0 \\
1 & 0.7 & 1 & 0 & 0 & 0.2 & 0 \\
0.3 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0.5 & 0 \\
0.8 & 0 & 0.2 & 0 & 0.5 & 1 & 0.9 \\
0.5 & 0 & 0 & 0 & 0 & 0.9 & 1
\end{array} \tag{7.2-10}
$$

In fuzzy information retrieval, any ***user's inquiry*** is expressed by a fuzzy set $I$ on $T$.

$$\begin{array}{cccccccc} & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 \\ \mathbf{I} = & [0 & 0.7 & 1 & 0.2 & 0 & 0 & 0] \end{array} \quad (7.2\text{-}11)$$

When the fuzzy relation $I$ is composed with the fuzzy relation $M$, an augmented inquiry, $A$, which may extend the original inquiry by associated index terms is obtained.

$$A = I \circ M \quad (7.2\text{-}12)$$

$$\begin{array}{cccccccc} & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 \\ \mathbf{A} = & [1 & 0.7 & 1 & 0.2 & 0 & 0.2 & 0] \end{array} \quad (7.2\text{-}13)$$

The retrieved documents are characterized by a fuzzy set $Q$ on $D$, which is obtained by composing the augmented inquiry $A$ with the relevance relation $R$.

$$Q = A \circ R \quad (7.2\text{-}14)$$

$$\begin{array}{cccccccccccccccc} & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 & d_9 & d_{10} & d_{11} & d_{12} & d_{13} & d_{14} & d_{15} \\ \mathbf{Q} = & [0.2 & 0 & 0.3 & 0 & 1 & 0.7 & 0.7 & 0.7 & 0 & 0.6 & 0 & 0 & 0.2 & 0 & 1] \end{array} \quad (7.2\text{-}15)$$

This set characterizes the degree to which each document matches with the user's interest as expressed by his or her inquiry.