Classical Control Theory

- ***Mechanism of feedback*** is the basic fundamental of ***control and automation***.
- By the mechanism of feedback, a mammal maintains its body temperature.
- The temperature in a building is kept to a desired setting even though the fluctuation of outside temperature.
- An aircraft can maintain its heading and altitude and can even land, all without human intervention, through feedback.
- Feedback is the mechanism that makes it possible for a biped to stand erect on two legs and to walk without falling.

1 Control Theory Background

- Feedback control engineering is regarded as the conscious, intentional use of the mechanism of feedback to control the behavior of a dynamic process.
- The history of control theory can be conveniently divided into three periods. The first, starting in prehistory and ending in the early 1940s, is termed as the ***primitive period***. This was followed by a ***classical period***, lasting 20 years, and finally came the ***modern period***.

Primitive Period

- The intentional use of feedback to improve the performance of dynamic system was started at around the beginning of the industrial revolution in the late 18th and early 19th centuries.
- The benchmark development was the ball-governor invented by James Watt to control the speed of his steam engine.

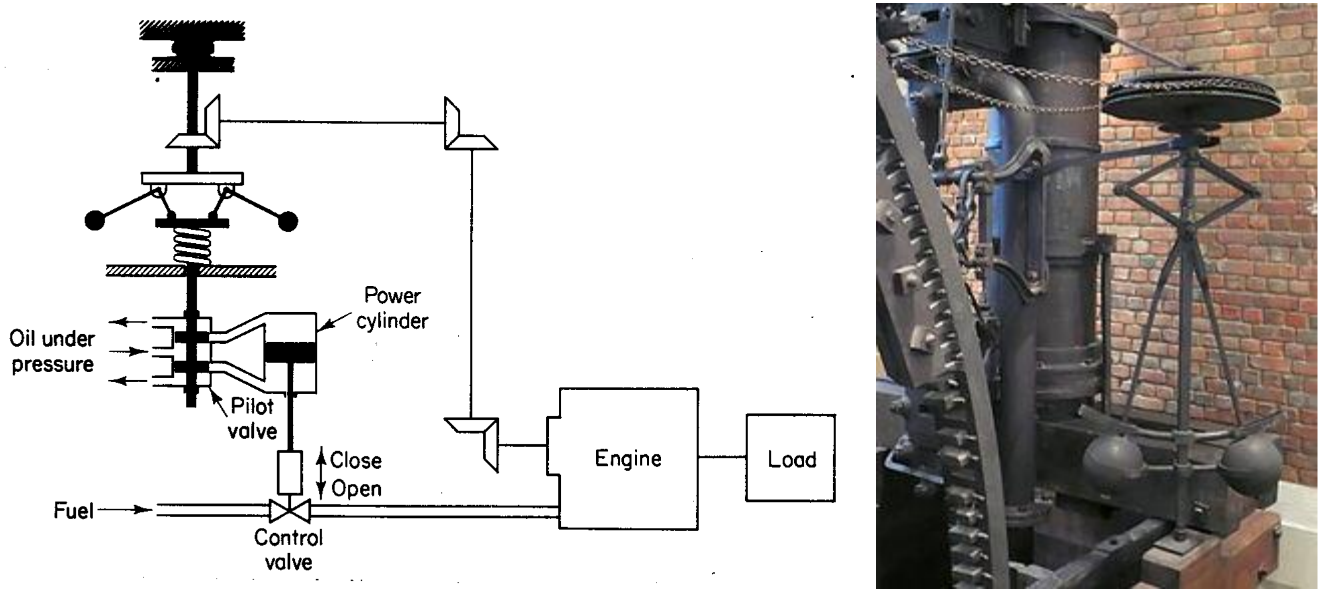


Figure 1-1 Watt's Speed Governor

- By the mid 19th century it was understood that the stability of a dynamic system was determined by the location of the roots of the algebraic characteristic equation. Routh invented the stability algorithm.

Classical Period

- The classical period of control theory begins during World War II in the Radiation Laboratory of MIT.
- Knowledge in the frequency response methods was developed by people such as Nyquist and Bode.
- Use of frequency-domain (Laplace transform) methods made possible the representation of a process by its transfer function.

Modern Period

- State-space methods are the cornerstone of modern control theory.
- In the modern approach the processes are characterized by systems of coupled, first-order differential equations.

2 Some Definitions in Control Theory

Static and Dynamic System:

- A ***dynamic element*** is one whose present output depends on past inputs.
- A ***static element*** is one whose output at any given time depends on only the input at that time.
- The present position of a car depends on where it started and what its velocity has been from the start, and thus it is a dynamic element.
- A resistor is a static element, because its present current depends on only the voltage applied at present, not on past voltages.

Lumped- and Distributed-Parameter Models:

- The dynamic model of a system that depends on location as well as time is called a ***distributed-parameter model***.
- The dynamic model of a system that depends only on time is called a ***lumped-parameter model***.

Example

Consider the temperature *T* of a metal plate. If the plate is heated at one side, the temperature will be a function of location and time, $T = T(t, x, y, z)$, and the model will have the form

$$f\left(T, \frac{\partial T}{\partial t}, \frac{\partial^2 T}{\partial x^2}, \frac{\partial^2 T}{\partial y^2}, \frac{\partial^2 T}{\partial z^2}\right) = 0 \tag{2-1}$$

But if the plate temperature is lumped with a single value, the model will have the form

$$f\,(T, \frac{dT}{dt}) = 0 \tag{2-2}$$

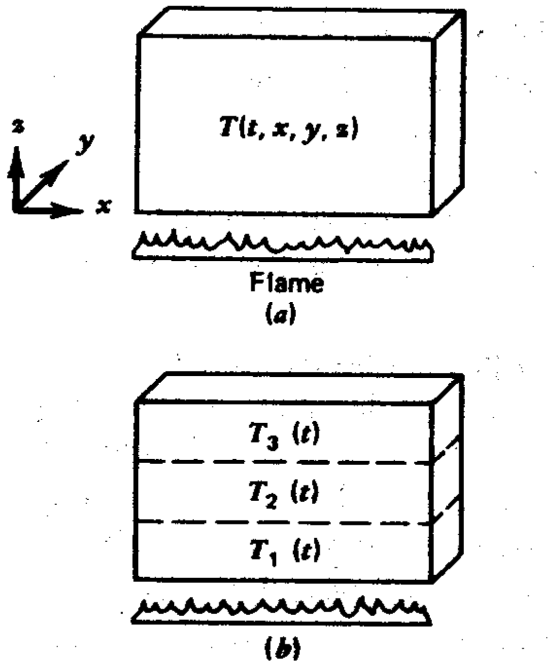


Figure 2-1 Temperature Distribution in a Plate (a) Distributed-Parameter Representation

(b) Lumped Parameter Representation Using Three Elements

Linear and Nonlinear Models:

Consider a system

$$y = f(x) \tag{2-3}$$

- The model is said to be ***linear*** if, for an input ax_1+bx_2, the output is

$$y = f(ax_1 + bx_2) = af(x_1) + bf(x_2) = ay_1 + by_2 \tag{2-4}$$

where a and b are arbitrary constants, x_1 and x_2 are arbitrary inputs, and

$$y_1 = f(x_1) \tag{2-5}$$

$$y_2 = f(x_2) \tag{2-6}$$

- The definition of linearity can be extended to include functions of more than one variables, such as $f(x,z)$. This function is linear if and only if

$$y = f(ax_1 + bx_2, az_1 + bz_2) = af(x_1, z_1) + bf(x_2, z_2) \tag{2-7}$$

- The linearity property is sometimes called the ***superposition principle***, because it states that a linear combination of inputs produces and output that is the superposition of the outputs that would be produced if each input term were applied separately.

Examples

$y = 5\ddot{x} + 4\dot{x} + 4x$: a linear system

$y = x^2$: a nonlinear system

Time- Variant and Invariant Models:

- Models with constant coefficients are called ***time-invariant*** or ***stationary models***.
- Models with variable coefficients are ***time-variant*** or ***nonstationary***.

Discrete- and Continuous-Time Models:

- ***Continuous-time model*** is represented by a differential equation. All the variables in the model are continuous.
- ***Discrete-time model*** is represented by a difference equation. All the variables in the model are discrete.

Model Order:

- ***Order of a model*** is the highest derivative order of a variable of that model.

Stochastic Models:

- If uncertainty in the values of the model's coefficients or inputs is great enough, it might justify using a ***stochastic model***.
- In stochastic model, the coefficients and inputs are described in terms of probability distributions involving, for example, their means and variances.

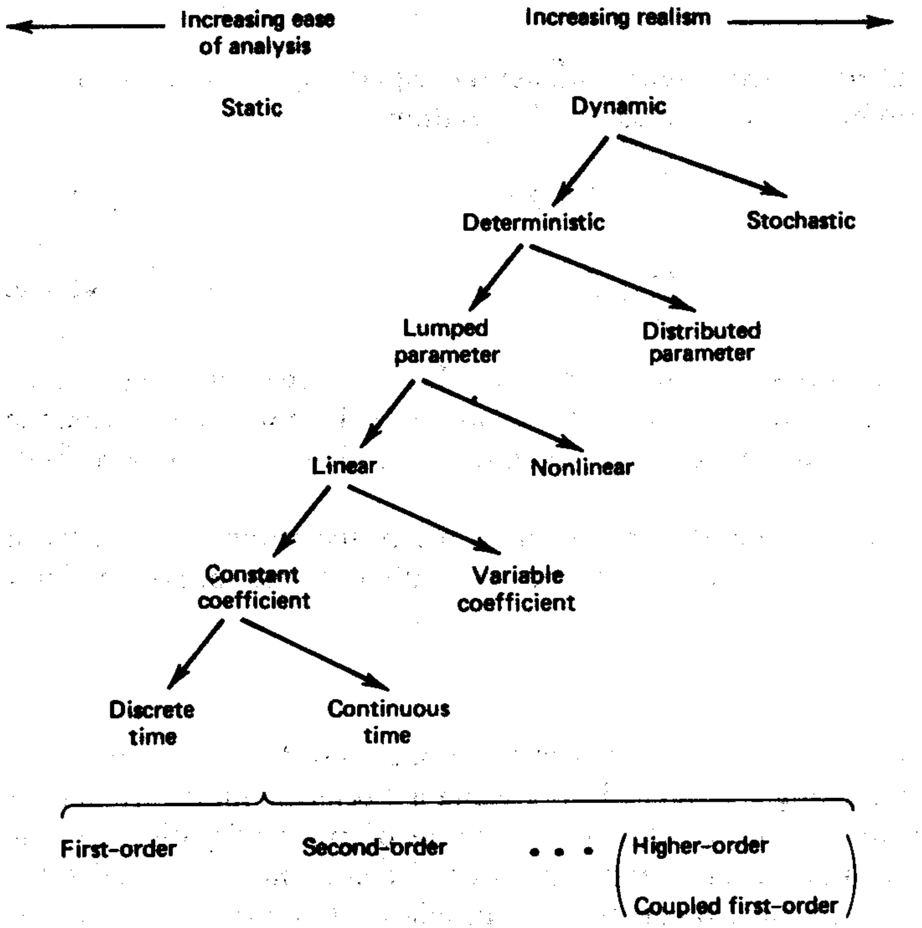


Figure 2-2 Classification of Mathematical Models

3 Linearization

- Because of the usefulness of the superposition principle, some nonlinear models are linearized to linear models.
- Consider the small angle approximation. If the angle of rotation θ of the lever is small, the rectilinear displacement of its ends is roughly proportional to θ such that $x = L\theta$.
- The same is not true for a large enough value of θ.

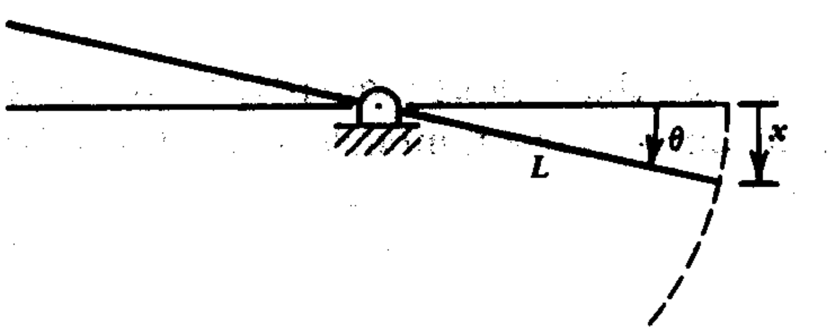


Figure 3-1 Small-Angle Approximation for the Displacement of a Lever Endpoint

- A systematic procedure based on the Taylor series expansion can be used for linearization.

$$w = f(y) \tag{3-1}$$

- A model that is approximately linear near the reference point (w_0, y_0) can be obtained by expanding $f(y)$ in a Taylor series near this point and truncating the series beyond the first-order term.

$$w = f(y) = f(y_0) + \left(\frac{df}{dy}\right)_0 (y - y_0) + \frac{1}{2!}\left(\frac{d^2 f}{dy^2}\right)_0 (y - y_0)^2 + \cdots \tag{3-2}$$

- If y is close enough to y_0, the terms involving $(y\text{-}y_0)^i$ for $i \geq 2$ are small compared to the first two terms.

$$w = f(y) \cong f(y_0) + \left(\frac{df}{dy}\right)_0 (y - y_0) \tag{3-3}$$

Let

$$m = \left(\frac{df}{dy}\right)_0 \tag{3-4}$$

$$z = w - w_0 = w - f(y_0) \tag{3-5}$$

$$x = y - y_0 \tag{3-6}$$

Then

$$z \cong mx \tag{3-7}$$

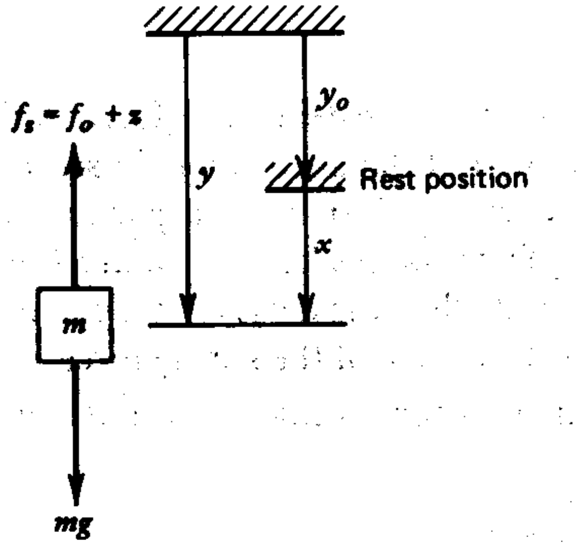


Figure 3-2 Free Body Diagram of the Mass-Spring System

- Consider a nonlinear spring, $f = y^2$.
- At the rest position y_0, the weight of the mass will equal the spring force, so that $mg = y_0^2$, or $y_0 = (mg)^{1/2}$.
- The Taylor series applied to the spring relation $f = y^2$ gives

$$f = y^2 \cong y_0^2 + \left(\frac{dy^2}{dy}\right)_0 (y - y_0) = y_0^2 + 2y_0(y - y_0) \qquad (3\text{-}8)$$

Let $z = f - f_0 = f - y_0^2$ and $x = y - y_0$. Thus,

$$z \cong 2y_0 x \tag{3-9}$$

- The Taylor series linearization technique can be extended to any number of variables.

$$w = f(y_1, y_2) \tag{3-10}$$

$$w \cong f(y_{10}, y_{20}) + \left(\frac{\partial f}{\partial y_1}\right)_0 (y_1 - y_{10}) + \left(\frac{\partial f}{\partial y_2}\right)_0 (y_2 - y_{20}) \tag{3-11}$$

Let

$$z = w - w_0 = w - f(y_{10}, y_{20}) \tag{3-12}$$

$$x_1 = y_1 - y_{10} \tag{3-13}$$

$$x_2 = y_2 - y_{20} \tag{3-14}$$

The linearized approximation is

$$z \cong \left(\frac{\partial f}{\partial y_1}\right)_0 x_1 + \left(\frac{\partial f}{\partial y_2}\right)_0 x_2 \tag{3-15}$$

The partial derivatives are the slopes of the function f in the y_1 and y_2 directions at the reference point.

4 Process Modeling

Mathematic model of a system can be determined by

- Experimental approach
  - Step response
  - Frequency response
  - Random response $\frac{y}{x} = \frac{a_{n-1}z^{-(n-1)} + a_{n-2}z^{-(n-2)} + \cdots + a_1 z^{-1} + a_0}{b_n z^{-n} + b_{n-1}z^{-(n-1)} + \cdots + b_1 z^{-1} + b_0} \Rightarrow$

$$a_{n-1}x(k-(n-1)) + a_{n-2}x(k-(n-2)) + \cdots + a_1 x(k-1) + a_0 x(k) = b_n y(k-n) + b_{n-1}y(k-(n-1)) + \cdots + b_1 y(k-1) + b_0 y(k)$$

- Theoretical approach

- If the model is too simplistic, the control system will not function as desired.
- If the model is too detailed, its complexity will hinder the application of design methods and prevent the designer from understanding the essential behavior of the system.

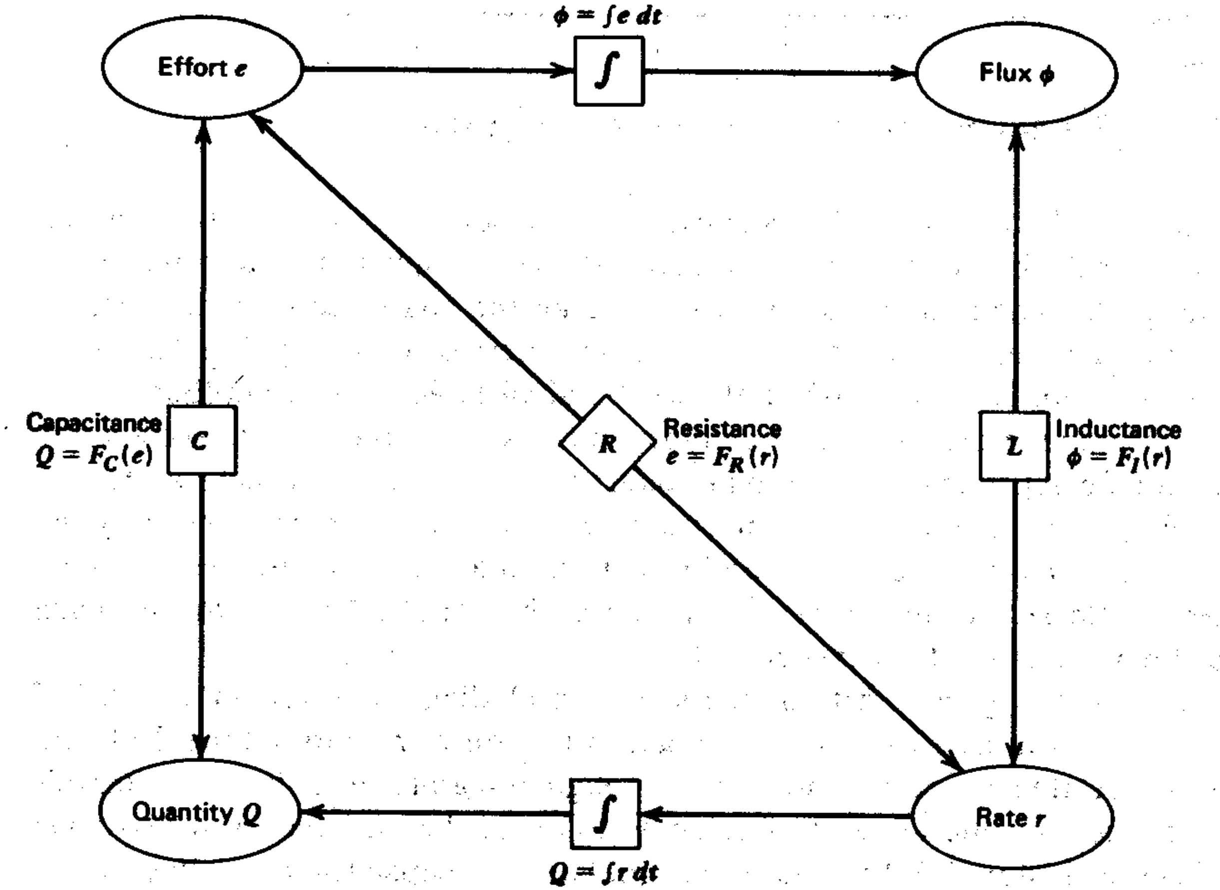


Figure 4-1 General Model Structure for an Arbitrary Physical System

Type	Rate r	Quantity $Q = \int r\,dt$	Effort e	Flux $\phi = \int e\,dt$
Electrical	Current i	Charge Q	Voltage v	Flux ϕ
Mechanical (translation)	Velocity v	Displacement x	Force f	Impulse M_x
Mechanical (rotation)	Angular velocity ω	Angular displacement θ	Torque T	Angular impulse M_θ
Fluid (incompressible)[a]	Mass flow rate q_m or volume flow rate q	Mass Q (or m) or volume V	Pressure p	None
Fluid (compressible)	Mass flow rate q_m	Mass Q (or m)	Pressure p	None
Thermal	Heat flow rate q_h	Heat energy Q_h	Temperature T	None

[a] Mass and volume are readily interchangeable for incompressible fluids; frequently, the symbol Q denotes volume and q mass flow rate.

Table 4-1 Primary System Variables and Their Usual Symbols

4.1 Electrical Systems

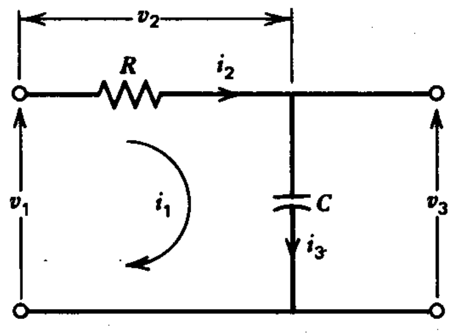


Figure 4.1-1 (a) A Series RC Circuit, (b) Its block diagram, (c) The Simplified Diagram

Conservation of energy: $v_1 = v_2 + v_3$ (a) Capacitor: $v_3 = Q/C$ (d)

Conservation of charge: $i_1 = i_2$ (b) Resistor: $i_2 = v_2/R$ (e)

$i_2 = i_3$ (c) Integral Causality: $Q = \int i_3 dt$ (f)

$$v_3 = \frac{1}{RC}\int (v_1 - v_3)dt$$

or

$$RC\frac{dv_3}{dt} + v_3 = v_1 \qquad (4.1\text{-}1)$$

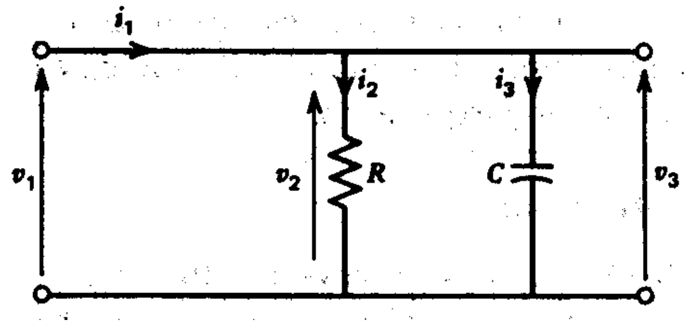


Figure 4.1-2 (a) A Parallel RC Circuit, (b) Its block diagram, (c) The Simplified Diagram

Conservation of energy:	$v_1 = v_2$	(a)	Capacitor:	$v_3 = Q/C$	(d)
	$v_2 = v_3$	(b)	Resistor:	$i_2 = v_2/R$	(e)
Conservation of charge:	$i_1 = i_2+i_3$	(c)	Integral Causality:	$Q = \int i_3 dt$	(f)

$$v_3 = \frac{1}{C}\int\left(i_1 - \frac{1}{R}v_3\right)dt$$

or

$$RC\frac{dv_3}{dt} + v_3 = Ri_1 \qquad (4.1\text{-}2)$$

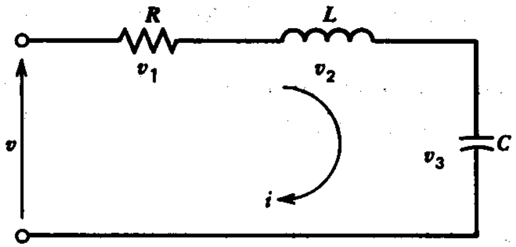


Figure 4.1-3 A Series RLC Circuit

Conservation of energy: $v = v_1+v_2+v_3$ (a) Integral Causality: $\phi = \int v\, dt$ (c)

Inductance: $\phi = Li$ (b)

$$v_1 = Ri$$

$$v_2 = L\frac{di}{dt}$$

$$v_3 = \frac{1}{C}\int i dt$$

$$v = Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt$$

or

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i = \frac{dv}{dt} \qquad (4.1\text{-}3)$$

4.2 Electromechanical Systems

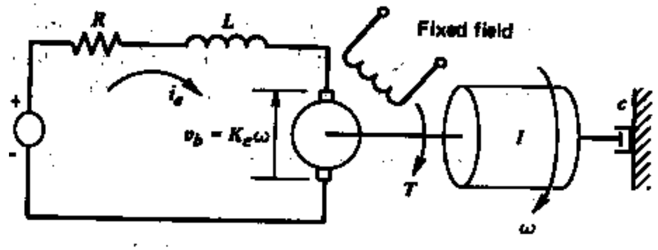


Figure 4.3-1Armature-Controlled DC Motor with a Load and Its Block Diagram

For armature-controlled motor, a torque T is proportional to the armature current i_a.

$$T = K_T i_a \tag{4.2-1}$$

The ***back emf*** (electromotive force) is proportional to the speed and is given by

$$e_b = K_e \omega \tag{4.2-2}$$

Kirchoff's voltage law gives

$$v = i_a R + L\frac{di_a}{dt} + K_e \omega \tag{4.2-3}$$

From Newton's law applied to the inertia I,

$$I\frac{d\omega}{dt} = K_T i_a - c\omega \tag{4.2-4}$$

5 Frequency-Domain Analysis

	$F(s)=\int_0^\infty e^{-st}f(t)dt$	$f(t)$
1.	1	$\delta(t)$
2.	$\frac{1}{s}$	1
3.	$\frac{n!}{s^{n+1}}$	t^n
4.	$\frac{1}{s+a}$	e^{-at}
5.	$\frac{b}{s^2+b^2}$	$\sin(bt)$
6.	$\frac{s}{s^2+b^2}$	$\cos(bt)$

Table 5-1 Laplace Transform Pairs

When $e^{ibt}=\cos(bt)+i\sin(bt)$

	$f(t)$	$F(s)=\int_0^\infty e^{-st} f(t)dt$
1.	$af_1(t)+bf_2(t)$	$aF_1(s)+bF_2(s)$
2.	$\dfrac{df}{dt}$	$sF(s)-f(0)$
3.	$\dfrac{d^2 f}{dt^2}$	$s^2F(s)-sf(0)-\dfrac{df}{dt}\Big\vert_{t=0}$
4.	$\dfrac{d^n f}{dt^n}$	$s^nF(s)-\sum_{k=1}^{n} s^{n-k}\dfrac{d^{k-1}f}{dt^{k-1}}\Big\vert_{t=0}$
5.	$\int f(t)dt$	$\dfrac{F(s)}{s}+\dfrac{\int f(t)dt\big\vert_{t=0}}{s}$
6.	$\begin{Bmatrix} 0 & ;t<D \\ f(t-D) & ;t\geq D \end{Bmatrix}$	$e^{-sD}F(s)$

Table 5-2 Properties of the Laplace Transform

$$y(s) = H(s)u(s) + f(y(0), \dot{y}(0), \cdots, u(0), \dot{u}(0), \cdots) \tag{5-1}$$

$$y(s) = H(s)u(s) \tag{5-2}$$

$$y(t) = \int_0^t H(t-\tau)u(\tau)d\tau \tag{5-3}$$

where $H(t)$ is the impulse-response of the system, and $H(s) = L[H(t)] = \int_0^\infty e^{-st} H(t)dt$

When the system of interest has the standard state-space representation

$$\dot{x} = Ax + Bu => sX - x(0) = AX + BU$$

$$y = Cx + Du => Y = AX + DU \tag{5-4}$$

$$H(s) = C(sI - A)^{-1}B + D = \frac{C(E_1 s^{k-1} + E_2 s^{k-2} + \cdots + E_k)B}{s^k + a_1 s^{k-1} + \cdots + a_k} + D \tag{5-5}$$

where the denominator of $H(s)$ is the characteristic polynomial

$$D(s) = |sI - A| = s^k + a_1 s^{k-1} + \cdots + a_k \tag{5-6}$$

and E_1, E_2, ..., E_k are the coefficient matrices of the adjoint matrix for the resolvent $(sI\text{-}A)^{-1}$. The root of the characteristic equation $|sI\text{-}A| = 0$ are called the ***characteristic roots*** or ***eigenvalues*** of the system.

- A transfer function $H(s)$ which is a proper rational function of s can be expanded in partial fractions

$$H(s) = \frac{N_1 s^{k-1} + \cdots + N_k}{s^k + d_1 s^{k-1} + \cdots + d_k} = H_1(s) + H_2(s) + \cdots + H_{\bar{k}}(s) \tag{5-7}$$

where
$$H_i(s) = \frac{R_{1i}}{s - s_i} + \frac{R_{2i}}{(s - s_i)^2} + \cdots + \frac{R_{v_i i}}{(s - s_i)^{v_i}} \tag{5-8}$$

The impulse response $H(t)$ of the system is given by the inverse Laplace transform

$$H(t) = H_1(t) + H_2(t) + \cdots + H_{\bar{k}}(t) \tag{5-9}$$

where
$$H_i(t) = (R_{1i} + R_{2i} t + \cdots + R_{v_i} t^{v_i - 1} / (v_i - 1)!) e^{s_i t} \tag{5-10}$$

- If the numerator of the transfer function is the same degree as the denominator, the constant term can be removed and the remainder written as a proper rational function.

$$H(s) = \frac{N_0 s^k + N_1 s^{k-1} + \cdots + N_k}{s^k + d_1 s^{k-1} + \cdots + d_k} = N_0 + \frac{\overline{N}_1 s^{k-1} + \cdots + \overline{N}_k}{s^k + d_1 s^{k-1} + \cdots + d_k} \tag{5-11}$$

$$\overline{N}_i = N_i - N_0 d_i \tag{5-12}$$

The corresponding impulse response has the form

$$H(t) = N_0 \delta(t) + \sum_{i=1}^{\bar{k}} (R_{1i} + \cdots + R_{v_i} t^{v_i - 1} / (v_i - 1)!) e^{s_i t} \tag{5-13}$$

where $\delta(t)$ is the unit impulse function.

- It is possible to conceive of systems having transfer functions in which the degree of the numerator is higher than the denominator. For example, an electrical inductor has the transfer function

$$\frac{v(s)}{i(s)} = z(s) = Ls \tag{5-14}$$

when the voltage $v(t)$ is regarded as the output and the current is regarded as the input. The impulse response of such systems, in general, contains not only impulses, but various derivatives of impulses.

5.1 Block Diagram Algebra

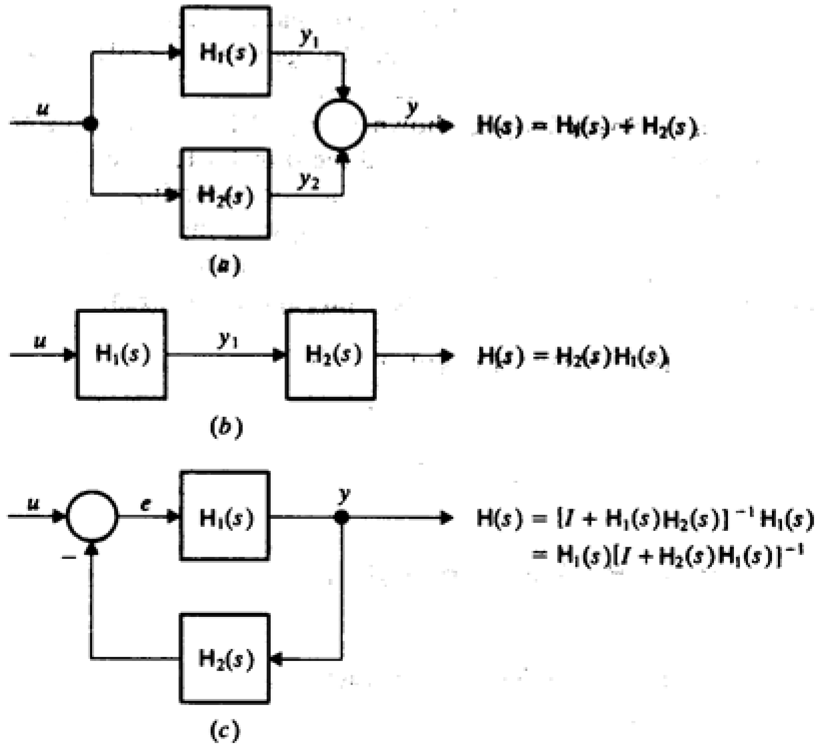


Figure 5.1-1 Subsystems in Combination (a) Subsystem in Parallel (b) Subsystem in Tandem (c) Single-Loop feedback system.

In figure 5.1-1(a),

$$y(s) = y_1(s) + y_2(s) = H_1(s)u(s) + H_2(s)u(s) = [H_1(s) + H_2(s)]u(s) \quad (5.1\text{-}1)$$

In figure 5.1-1(b),

$$y_1(s) = H_1(s)u(s) \quad (5.1\text{-}2)$$

and

$$y(s) = H_2(s)y_1(s) \quad (5.1\text{-}3)$$

Thus

$$y(s) = H_2(s)H_1(s)u(s) \quad (5.1\text{-}4)$$

In figure 5.1-1(c),

$$y(s) = H_1(s)e(s) = H_1(s)[u(s) - z(s)] \quad (5.1\text{-}5)$$

But

$$z(s) = H_2(s)y(s) \quad (5.1\text{-}6)$$

Thus

$$y(s) = H_1(s)[u(s) - H_2(s)y(s)] \quad (5.1\text{-}7)$$

or

$$[I + H_1(s)H_2(s)]y(s) = H_1(s)u(s) \quad (5.1\text{-}8)$$

and finally

$$y(s) = [I + H_1(s)H_2(s)]^{-1}H_1(s)u(s) \quad (5.1\text{-}9)$$

Thus, the transfer function (matrix) of the system containing a feedback loop is

$$H(s) = [I + H_1(s)H_2(s)]^{-1}H_1(s) \quad (5.1\text{-}10)$$

The matrix $I + H_1(s)H_2(s)$ is called return-difference matrix. These values of s are the pole of the system.

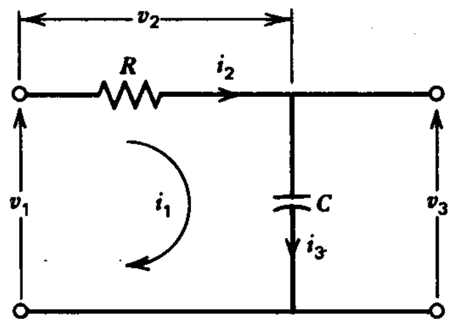


Figure 5.1-2 (a) A Series RC Circuit, (b) Its block diagram, (c) The Simplified Diagram

$$RC\frac{dv_3}{dt} + v_3 = v_1 \tag{5.1-11}$$

$$\frac{dv_3}{dt} = \frac{1}{RC}(v_1 - v_3) \tag{5.1-12}$$

$$sV_3 - v_3(0) = \frac{1}{RC}(V_1 - V_3) \tag{5.1-13}$$

$$\frac{V_3}{V_1} = \frac{1}{RCs + 1} \tag{5.1-14}$$

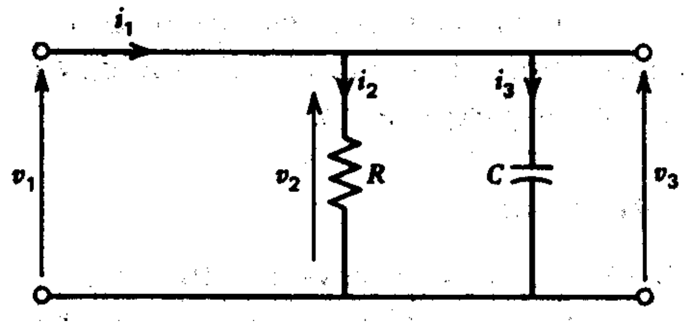


Figure 5.1-3 (a) A Parallel RC Circuit, (b) Its block diagram, (c) The Simplified Diagram

$$RC\frac{dv_3}{dt} + v_3 = Ri_1 \tag{5.1-15}$$

$$\frac{dv_3}{dt} = \frac{1}{C}(i_1 - \frac{v_3}{R}) \tag{5.1-16}$$

$$sV_3 - v_3(0) = \frac{1}{C}(I_1 - \frac{V_3}{R}) \tag{5.1-17}$$

$$\frac{V_3}{I_1} = \frac{R}{RCs+1} \tag{5.1-18}$$

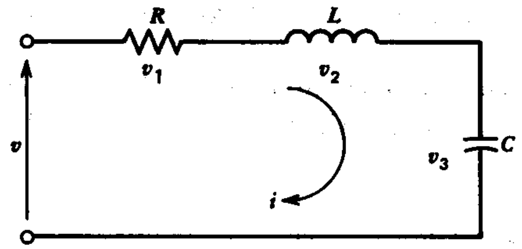


Figure 5.1-4 A Series RLC Circuit

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dv}{dt} \qquad (5.1\text{-}19)$$

$$L(s^2 I - si(0) - \frac{di(0)}{dt}) + R(sI - i(0)) + \frac{1}{C}I = sV - v(0) \qquad (5.1\text{-}20)$$

$$\frac{I}{V} = \frac{s}{Ls^2 + Rs + 1/C} \qquad (5.1\text{-}21)$$

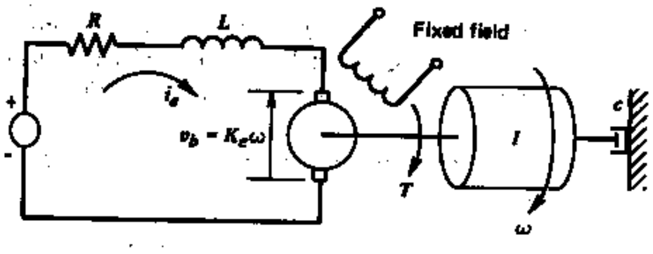


Figure 5.1-5 Armature-Controlled DC Motor with a Load and Its Block Diagram

Kirchoff's voltage law gives

$$v = i_a R + L\frac{di_a}{dt} + K_e\omega \tag{5.1-21}$$

$$V = I_a R + L(sI_a - i(0)) + K_e\Omega \tag{5.1.22}$$

$$I_a = \frac{V - K_e\Omega}{Ls + R} \tag{5.1.23}$$

From Newton's law applied to the inertia *I*,

$$I\frac{d\omega}{dt} = K_T i_a - c\omega \tag{5.1-24}$$

$$I(s\Omega - \omega(0)) = K_T I_a - c\Omega \tag{5.1-25}$$

$$\frac{\Omega}{V} = \frac{K_T}{LIs^2 + (Lc + IR)s + (Rc + K_T K_e)} \tag{5.1-26}$$

5.2 Stability

- Stability: the ability of the system to operate under a variety of conditions without self-destructing.
- The basic stability criterion of linear time-invariant systems is directly determined by the locations of the system poles, the roots of the characteristic equation of the system.

Ability of a system to return to equilibrium relates to the unforced system

$$\dot{x} = Ax \tag{5.2-1}$$

$$sX - x_0 = AX \tag{5.2.2}$$

$$(sI - A)X = x_0 \tag{5.2.3}$$

$$X = (sI - A)^{-1} x_0 \tag{5.2.4}$$

$$x(t) = e^{At} x_0 \tag{5.2-5}$$

where e^{At} is the state-transition matrix, given by

$$e^{At} = L^{-1}[(sI\text{-}A)^{-1}] = \sum_{i=1}^{k} \left(\sum_{j=1}^{v_i} \frac{R_{ji} t^j}{j!} \right) e^{s_i t} \tag{5.2-6}$$

Examples: $s_i = 5; \lim_{t\to\infty}(e^{5t}) = \infty$, $s_i = -5; \lim_{t\to\infty}(e^{-5t}) = 0$, $s_i = 0; \lim_{t\to\infty}(e^{0t}) = 1$

The following properties can be directly inferred from the form of the state-transition matrix.

1. If the real parts of all the characteristic roots are strictly negative, then e^{At} tends asymptotically to zero. Hence, no matter how large the initial state x_0 is, $x(t)\to 0$ as $t\to\infty$. The system is said to be ***asymptotically stable***.
2. If any characteristic root has a strictly positive real part, the state-transition matrix will have at least one term which will tend to infinity as $t\to\infty$. In this case it is always possible to find some initial state which will cause $x(t)$ to become infinite. The system is said to be ***unstable***.
3. When all the characteristic roots have nonpositive real parts, but one or more of the characteristic roots has a zero real part., if all the characteristics roots having zero real parts are simple root, then the corresponding terms in the state-transition matrix are of the form

$$R_i e^{jw_i t} \tag{5.2-7}$$

The solution is bounded but will not approach zero asymptotically. This type of system is said to be ***stable but not asymptotically stable***. But if the zero real part is a repeated root, then owing to the polynomial in t that multiplies $e^{jw_i t}$, there will be at least one term in e^{At} which will tend to infinity as $t\to\infty$.

Condition		Implication
1. $\mathrm{Re}(s_i) < 0$	for all i	System is asymptotically stable
2. $\mathrm{Re}(s_i) > 0$	for some i	System is unstable
3. $\mathrm{Re}(s_i) = 0$	for some $i = j$, and	
(*a*) s_i is simple root for all such j		System is stable, but not asymptotically stable
(*b*) s_j is multiple root for some such j		System is unstable

Table 5.2-1 Stability Conditions for Linear Systems

- There are two methods developed by E.J. Routh and Hurwitz that do not require actual calculation of the roots to determine stability of the system.

The characteristic polynomial of the system to be tested for stability is assumed to be of the form

$$D(s) = s^k + a_1 s^{k-1} + \cdots + a_{k-1} s + a_k \quad (5.2\text{-}8)$$

	1	a_2	a_4	$a_6 \cdots$
	a_1	a_3	a_5	$a_7 \cdots$
$\alpha_1 = \frac{1}{a_1}$	$b_1 = a_2 - \alpha_1 a_3$	$b_2 = a_4 - \alpha_1 a_5$	$b_3 = a_6 - \alpha_1 a_7$	$\cdots\cdots$
$\alpha_2 = \frac{a_1}{b_1}$	$c_1 = a_3 - \alpha_2 b_2$	$c_2 = a_5 - \alpha_2 b_3$	$\cdots\cdots\cdots\cdots$	
$\alpha_3 = \frac{b_1}{c_1}$	$d_1 = b_2 - \alpha_3 c_2$	$\cdots\cdots\cdots\cdots$		
$\alpha_4 = \frac{c_1}{d_1}$	$\cdots\cdots\cdots\cdots$			
$\vdots$				

Table 5.2-2 Routh Table

- The theorem of the Routh algorithm is that the roots of $D(s) = 0$ lie in the left half-plane, excluding the imaginary axis, if and only if all the α's are strictly positive.
- In the Hurwitz's stability algorithm, the roots of $D(s) = 0$ lie in the left half-plane if and only if all the determinants of Hurwitz matrices, 1 to k square matrices, are positive.
- The Hurwitz's stability algorithm is similar to Routh's so we can combine them and recall it as Routh-Hurwitz stability algorithm.

$$H = \underbrace{\begin{bmatrix} a_1 & a_3 & \cdots & \cdots \\ 1 & a_2 & \cdots & \cdots \\ \hline 0 & a_1 & a_3 & \cdots \\ 0 & 1 & a_2 & \cdots \\ \hline 0 & 0 & a_1 & \cdots \\ 0 & 0 & 1 & \cdots \end{bmatrix}}_{k \text{ columns}} \Bigg\updownarrow k \text{ rows}$$

Table 5.2-3 Hurwitz's Matrix

Examples

(a) $s^4+2s^3+3s^2+4s+5=(s-0.29+1.42j)(s-0.29-1.42j)(s+1.29+0.86j)(s+1.29-0.86j)=0$

		1	3	5
	α	2	4	0
1	1/2	3-(1/2)4=1	5-(1/2)0=5	0
2	2/1=2	4-(2)5=-6	0	0
3	-1/6	5+(1/6)0=5	0	0
4	-6/5	0	0	0

Since α_3 and α_4 are negative, this system is unstable. The free response follows $A_1e^{0.29t}\cos(1.42t+\phi_1)+A_2e^{-1.29t}\cos(0.86t+\phi_2)$.

(b) $s^3+2s^2+s+2=(s+2)(s+j)(s-j)=0$

		1	1
	α	2	2
1	1/2	1-(1/2)2=0,ε	0
2	2/ε	2-(2/ε)0=2	0
3	ε/2	0	0

Since there is no negative α but α_3 is zero, this system is stable but not asymptotically stable. It has a pair of imaginary roots. The free response follows $A_1e^{-2t}+A_2\cos(t+\phi_2)$.

(c) $s^3 - 3s + 2 = (s+2)(s-1)(s-1) = 0$

		1	-3
	α	0,ε	2
1	1/ε	-3-(1/ε)2	0
2	-ε/(3+(1/ε)2)	2-(α)0=2	0
3	-(3+(1/ε)2)/2	0	0

Since α_2 is zero and α_3 is negative, this system is unstable. The free response follows $A_1e^{-2t} + A_2e^{t} + A_3te^{t}$.

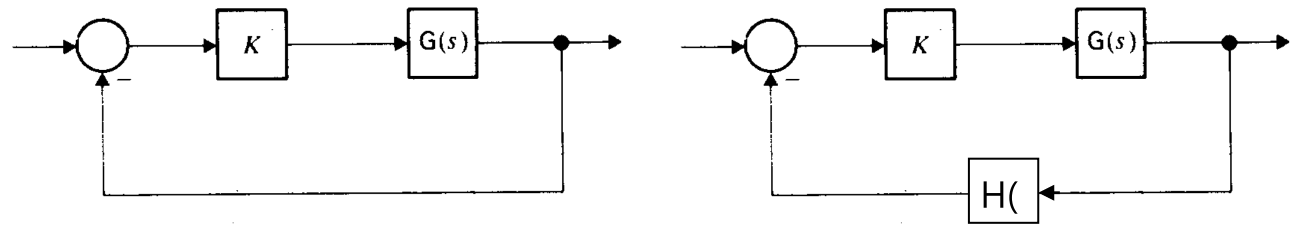
(d) $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = (s+2)(s+1)(s-1)(s+5j)(s-5j) = 0$

		1	24	-25
	α	2	48	-50
1	1/2	24-(1/2)48=0, ε	-25+(1/2)50=0	0
2	2/ε	48	-50	0
3	ε/48	50ε/48	0	0
4	1304/(50ε)	-50	0	0
5	-ε /48	0	0	0

Since α_5 is negative, this system is unstable. It has a positive real part and a pair of imaginary roots. The free response follows $A_1e^{-2t} + A_2e^{-t} + A_3e^{t} + A_4\cos(5t + \phi_4)$.

5.3 Graphical Methods

5.3.1 Root-Locus Method



(a) (b)

Figure 5.3.1-1 Single-Loop Feedback System

- The return-difference function of the diagram,

$$T(s) = 1 + KG(s)\,,\ T(s) = 1 + KG(s)H(s) \qquad (5.3.1\text{-}1)$$

- Open-loop transfer function of the figure,

$$T_o(s) = KG(s)\,,\ T_o(s) = KG(s)H(s) \qquad (5.3.1\text{-}2)$$

- Closed-loop transfer function,

$$T_c(s) = \frac{KG(s)}{1 + KG(s)}\,,\ T_c(s) = \frac{KG(s)}{1 + KG(s)H(s)} \qquad (5.3.1\text{-}3)$$

- The root-locus method, developed by Evans in 1948 is simply a plot of the locations in the complex plane of the roots of $T(s) = 0$, as the loop gain is varied.
- The open-loop transfer function is assumed to be a rational function of s,

$$T_o(s) = KC\frac{\prod^{n_z}{}_{i=1}(s - z_i)}{\prod^{n_p}{}_{i=1}(s - p_i)} \tag{5.3.1-4}$$

where C is a real constant, z_i (i =1, ..., n_z) are the open-loop zeros and p_i (i = 1, ..., n_p) are open-loop poles.

- The return-difference, which is the characteristic equation,

$$T(s) = 1 + KC\frac{\prod^{n_z}{}_{i=1}(s - z_i)}{\prod^{n_p}{}_{i=1}(s - p_i)} = \prod^{n_p}{}_{i=1}(s - p_i) + KC\prod^{n_z}{}_{i=1}(s - z_i) \tag{5.3.1-5}$$

- It is seen that as $K\rightarrow 0$, the closed-loop poles tend to the open-loop poles p_i.
- On the other hand, as the gain K tends to infinity, the closed-loop poles tend to the open-loop zeros.
- If $G(s)$ is a proper rational function, there are fewer open-loop zeros than open-loop poles. Since the number of closed-loop poles does not change as K is varied, so the excess numbers of the open-loop poles over zeros roots go to infinity.

$$1 + K\frac{1}{s^e} = 0 \text{ and } e = n_p - n_z \tag{5.3.1-6}$$

$$s^e + K = 0 \tag{5.3.1-7}$$

$$\angle(s^e) = e\angle(s) = \angle(-K) = -n180°;\ n = \pm 1, \pm 3, \cdots \tag{5.3.1-8}$$

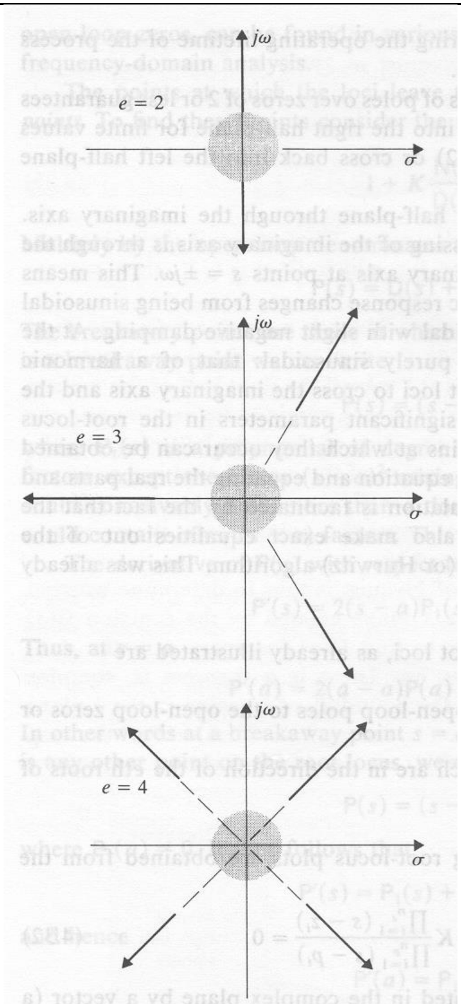


Figure 5.3.1-2 Asymptotes of Root Loci for Several Values of Excess Poles

- The points at which the loci leave the real axis are known as ***breakaway points***. The breakaway points are those at which $P(s)$ has a multiple root. Thus if $s = a$ is a breakaway point, derivative of $P(s)$ is zero.

$$P(s) = (s-a)^2 P_1(s) \tag{5.3.1-9}$$

$$\frac{dP(s)}{ds} = P'(s) = 2(s-a)P_1(s) + (s-a)^2 P_1'(s) \tag{5.3.1-10}$$

- If $s = a$ is any other point on the root locus,

$$P(s) = (s-a)P_1(s) \tag{5.3.1-11}$$

$$P'(s) = P_1(s) + (s-a)P_1'(s) \tag{5.3.1-12}$$

- The return difference at breakaway points,

$$1 + K\frac{A}{B} = 0; K = -\frac{B}{A} \tag{5.3.1-13}$$

$$P = B + KA = 0 \tag{5.3.1-14}$$

$$P' = B' + KA' = 0; K = -\frac{B'}{A'} \tag{5.3.1-15}$$

$$P = B - \frac{B'}{A'}A = 0 \tag{5.3.1-16}$$

$$BA' - B'A = 0 \tag{5.3.1-17}$$

$$K' = -\frac{B'A - BA'}{A^2} = 0 \tag{5.3.1-18}$$

Standard Form: $1 + KP(s) = 0;\ K \geq 0$

$$P(s) = N(s)/D(s);\ N(s) = s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0;\ P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0;\ m \leq n$$

Terminology: The zeros of $P(s)$ are the roots of $N(s) = 0$. The poles of $P(s)$ are the roots of $D(s) = 0$.

Angle Criterion: $\angle P(s) = n180°;\ n = \pm 1, \pm 3, \cdots$

Magnitude Criterion: $K = 1/|P(s)|$

Plotting Guides:

Guide 1: The root-locus plot is symmetric about the real axis.

Guide 2: The number of loci equals the number of poles of $P(s)$.

Guide 3: The loci start at the poles of $P(s)$ with $K = 0$, and terminate with $K = \infty$ either at the zeros of $P(s)$ or at infinity.

Guide 4: The root locus can exist on the real axis only to the left of an odd number of real poles and/or zeros.

Guide 5: The locations of breakaway and break-in points are found by determining where the parameter K attains a local maximum or minimum on the real axis.

Table 5.3.1-1 Plotting Guides for the Primary Root Locus

Guide 6: The loci that do not terminate at a zero approach infinity along asymptotes. The angles that the asymptotes make with the real axis are found from $\theta = \dfrac{n180^\circ}{Z - P}$; $n = \pm 1, \pm 3, \cdots$

Guide 7: The asymptotes intersect the real axis at the common point $\sigma = \dfrac{\Sigma s_p - \Sigma s_z}{Z - P}$

Guide 8: The points at which the loci cross the imaginary axis and the associated values of K can be found with the Routh-Hurwitz criterion or by substituting $s = i\omega$ into the equation. The frequency ω is the crossover frequency.

Guide 9: Angles of departure and angles of arrival are determined by choosing an arbitrary point infinitely close to the pole or zero in question and applying the angle criterion.

Guide 10: For the polynomial equation, $s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$, the sum of the roots $r_1 + r_2 + \cdots + r_n = -a_{n-1}$.

Guide 11: Once the root locus is drawn, it is scaled with the magnitude criterion.

Table 5.3.1-2 Plotting Guides for the Primary Root Locus (cont')

- Characteristic equation of the second order system,

$$As^2 + Bs + C = 0\,;\; s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad (5.3.1\text{-}19)$$

$$s = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} \quad (5.3.1\text{-}20)$$

- The free response,

$$Ae^{-\zeta\omega_n t}\cos((\omega_n\sqrt{1-\zeta^2})t + \phi \quad (5.3.1\text{-}21)$$

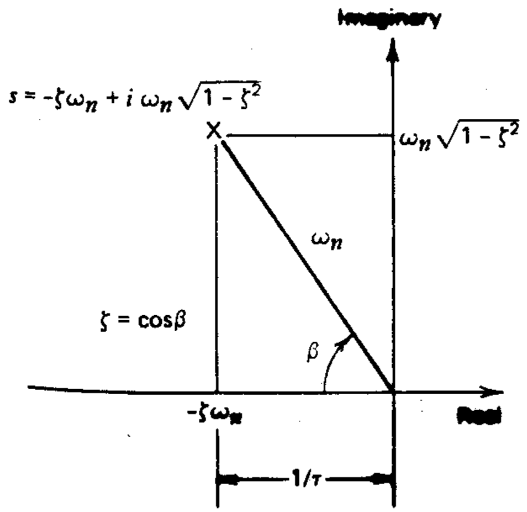


Figure 5.3.1-3 Location of the Upper Complex Root in Terms of the Parameters $\zeta, \tau, \omega_n, \omega_d$

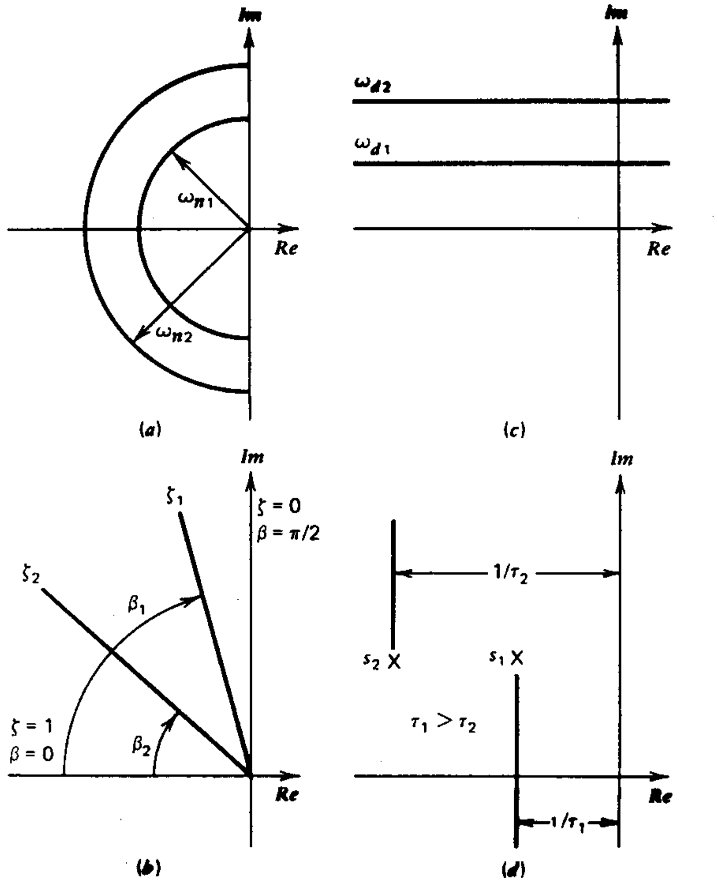
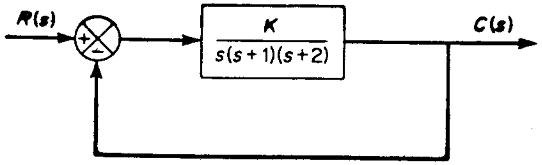
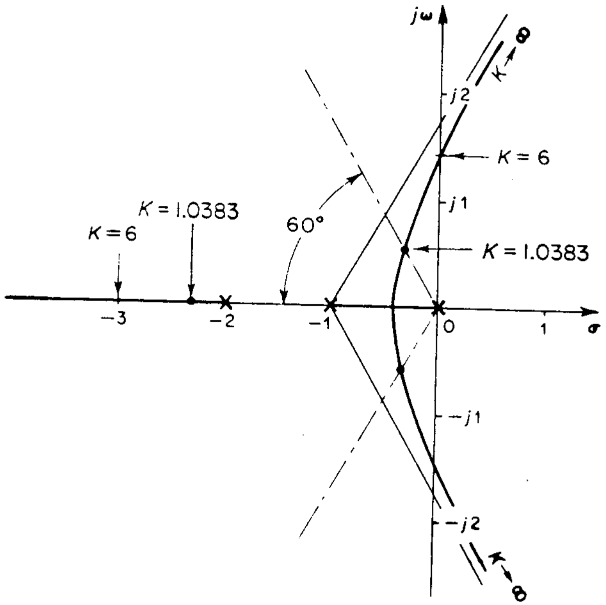


Figure 5.3.1-4 Graphical Representation of the Parameters $\zeta, \tau, \omega_n, \omega_d$ in the Complex Plane

$$G(s) = \frac{K}{s(s+1)(s+2)}, \quad H(s) = 1$$





1. P = 0, -1, -2
2. $Z = \phi$
3. e = 3

   $\angle s = -n180/3 = -n\,60 = \pm 60°, \pm 180°$

4. $\sigma = [(0-1-2)-(0)]/3 = -1$
5. $T_c = \dfrac{K}{s(s+1)(s+2)+K}$

   $P = s^3 + 3s^2 + 2s + K$

   $P' = 3s^2 + 6s + 2 = 0$

   $s = -1 \pm 1/\sqrt{3} \rightarrow -1 + 1/\sqrt{3}$

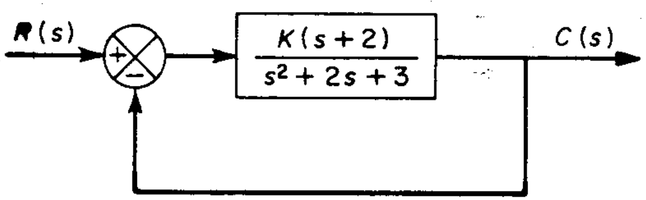
   $K = 0.385$

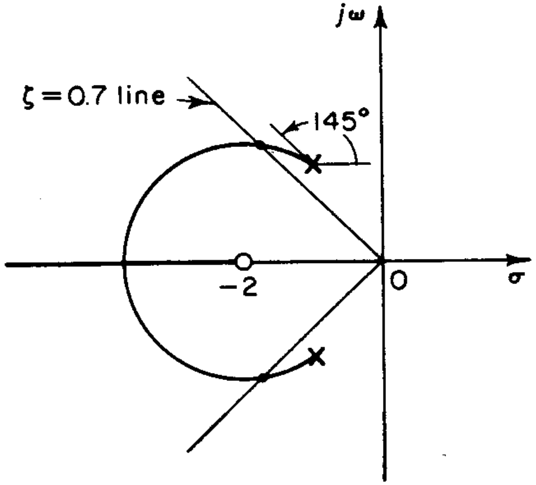
6. $P(j\omega) = -j\omega^3 - 3\omega^2 + 2\omega j + K = 0$

   $-j\omega^3 + 2\omega j = 0 \rightarrow \omega = \pm\sqrt{2}$

   $-3\omega^2 + K = 0 \rightarrow K = 6$

$$G(s) = \frac{K(s+2)}{s^2 + 2s + 3}, \qquad H(s) = 1$$





1. $P = -1 \pm \sqrt{2}\,j$

2. $Z = -2$

3. $e = 1$

$\angle s = -n180/1 = -n\,180 = \pm 180°$

4. $\sigma = [(-2)-(-2)]/1 = 0$

5. $T_c = \dfrac{K(s+2)}{s^2 + (2+K)s + (3+2K)}$

$P = s^2 + (2+K)s + (3+2K)$

$P' = 2s + 2 + K = 0$

$s = \dfrac{-(2+K) \pm \sqrt{(2+K)^2 - 4(3+2K)}}{2} \rightarrow$ real number

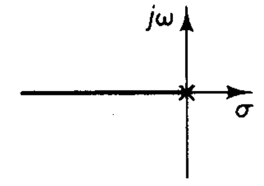
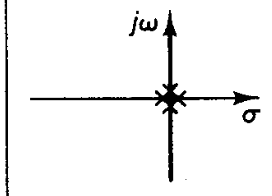
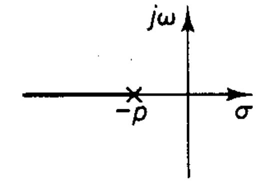
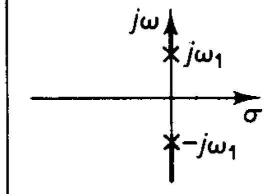
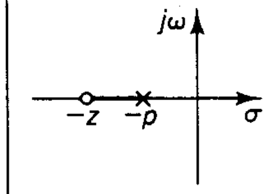
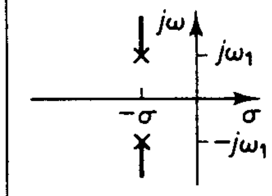
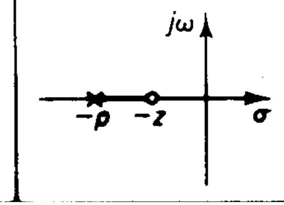
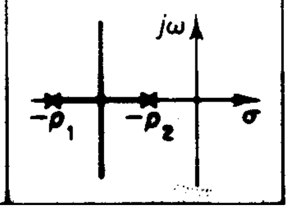
$(2+K)^2 - 4(3+2K) = 0 \rightarrow K = 2 \pm 2\sqrt{3} \rightarrow 2 + 2\sqrt{3}$

$s = -(2+\sqrt{3})$

6. $P(j\omega) = -\omega^2 + (2+K)\omega j + (3+2K) = 0$

$(2+K)\omega = 0 \rightarrow \omega = 0$

$-\omega^2 + (3+2K) = 0 \rightarrow K = -3/2 \rightarrow$ no crossover point

$G(s)H(s)$	Open-loop Pole-zero Locations and Root Loci	$G(s)H(s)$	Open-loop Pole-zero Locations and Root Loci
$\frac{K}{s}$		$\frac{K}{s^2}$	
$\frac{K}{s+p}$		$\frac{K}{s^2+\omega_1^2}$	
$\frac{K(s+z)}{s+p}$ $(z > p)$		$\frac{K}{(s+\sigma)^2+\omega_1^2}$	
$\frac{K(s+z)}{s+p}$ $(z < p)$.		$\frac{K}{(s+p_1)(s+p_2)}$	

5.3.2 Nyquist Diagram

- The earliest graphical method investigating the stability of linear systems was developed by H. Nyquist in 1932 and is based on the polar plot of the loop transmission transfer function.

$$1 + KG(s) = 0 \quad \text{or} \quad G(s) = -\frac{1}{K} \quad \text{or} \quad KG(s) = -1 \tag{5.3.1-1}$$

- If s locates in the right half of the s plane, the system is unstable. For every s in the right half-plane, there is a point $z = G(s)$ or $KG(s)$ in the z plane.
- If the region of the z plane that is the map of the right half of the s plane under the function $G(s)$ covers the point $-1/K$ (or $KG(s)$ covers the point -1), the system is unstable; if the map of $G(s)$ does not cover the point $-1/K$ (or $KG(s)$ does not cover the point -1), the system is stable.
- The quantitative distance between the point $-1/K$ and the map of the right half-plane on real axis is the ***gain margin*** of the system.

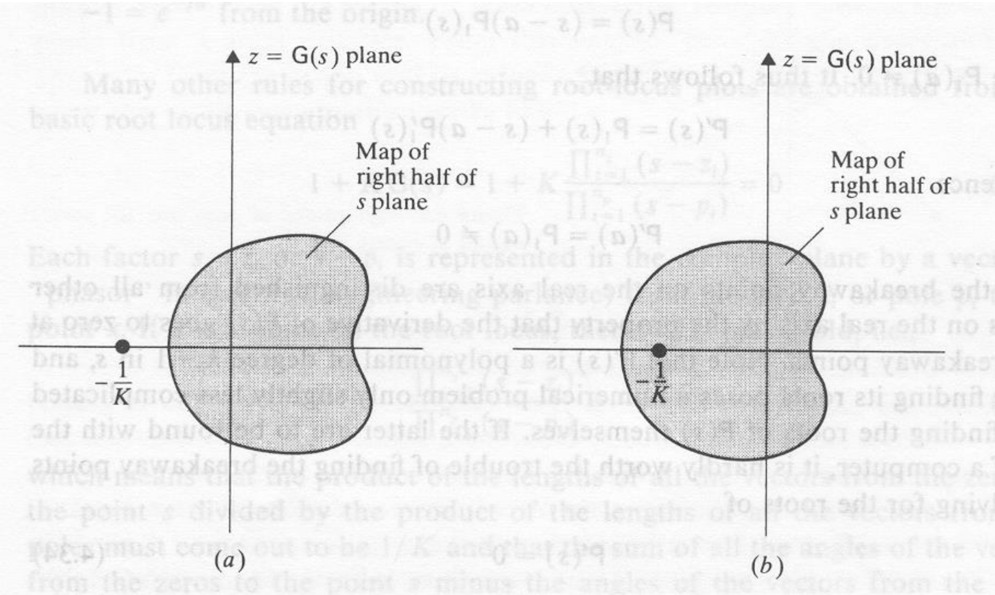


Figure 5.3.2-1 Graph in *z* Plane of the Right Half-Plane in *s* Plane

(a) Stable (b) Unstable System

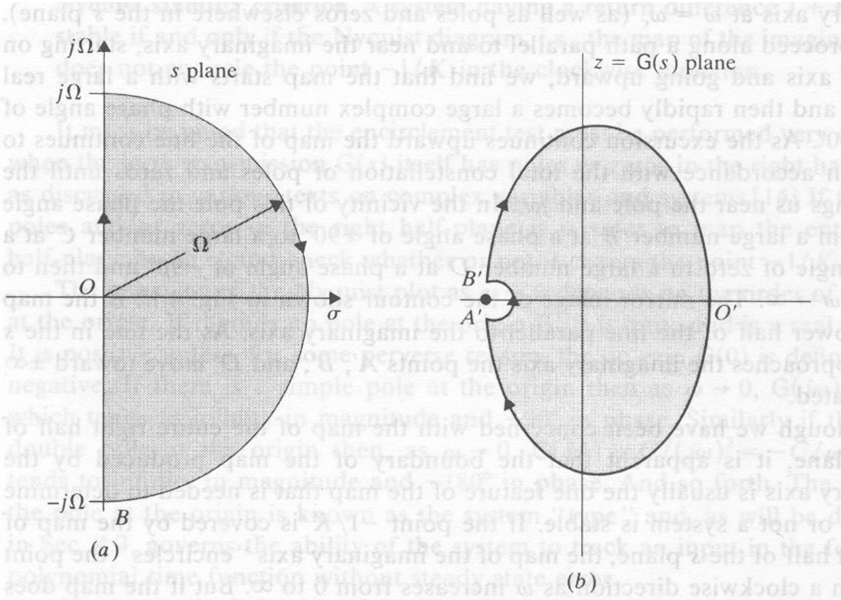
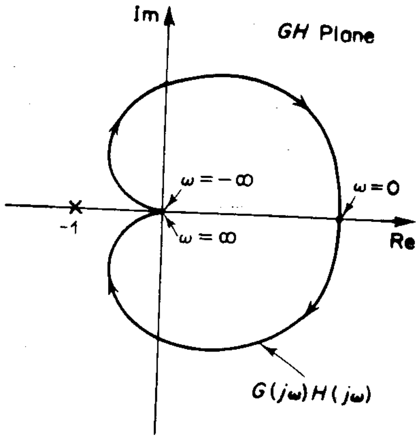


Figure 5.3.2-2 Method to Map Right Half of *s* Plane into *z* Plane

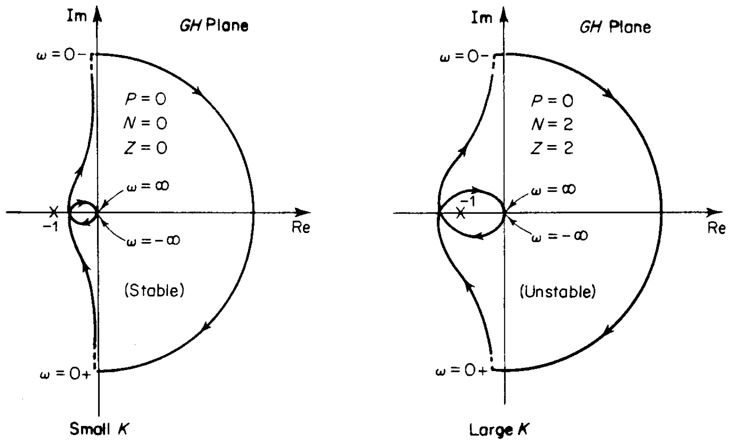
(a) Semicircle Approximation of All Right Half-Plane (b) Map of Semicircle

$$G(s)H(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)}$$



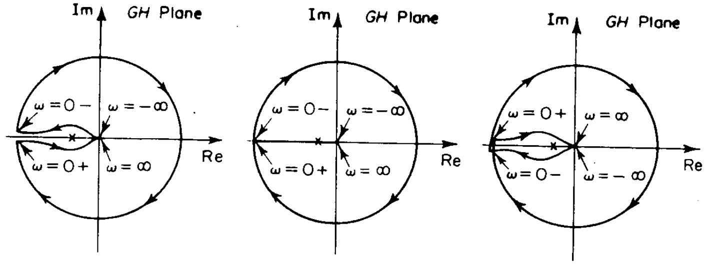
Point	s	z
O	$0,+0j$	K
A	$j\omega$	$\frac{K}{(1-T_1T_2\omega^2)+(T_1+T_2)\omega j} = \frac{K[(1-T_1T_2\omega^2)-(T_1+T_2)\omega j]}{(1-T_1T_2\omega^2)^2+((T_1+T_2)\omega)^2}$
	Intersection point	$(0,-\frac{K\sqrt{T_1T_2}}{T_1+T_2})$ -> $\omega = \frac{1}{\sqrt{T_1T_2}}$
	$\omega \to \infty$	$-0-0j$ stable

$$G(s) = \frac{K}{s(T_1 s + 1)(T_2 s + 1)}$$



Point	s	z
O	$0,+0j$	$\frac{K}{s}$ -> $\infty, -\infty j$
A	$j\omega$	$\frac{K}{-(T_1+T_2)\omega^2 + (\omega - T_1T_2\omega^3)j} = \frac{K[-(T_1+T_2)\omega^2 - (\omega - T_1T_2\omega^3)j]}{((T_1+T_2)\omega^2)^2 + (\omega - T_1T_2\omega^3)^2}$
	Intersection point	$(-\frac{KT_1T_2}{T_1+T_2},0)$ -> $\omega = \frac{1}{\sqrt{T_1T_2}}$, max $K = \frac{T_1+T_2}{T_1T_2}$
	$\omega \to \infty$	$-0+0j$

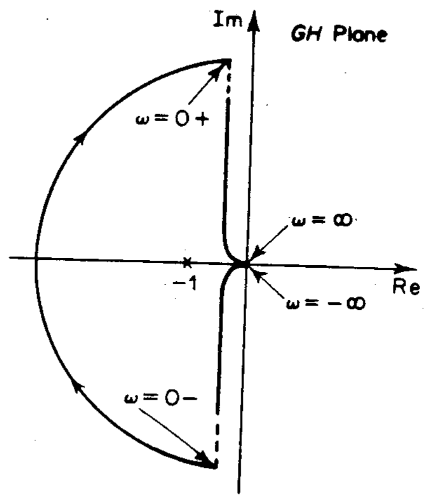
$$G(s)H(s) = \frac{K(T_2 s + 1)}{s^2(T_1 s + 1)}$$



$T_1 < T_2$ (Stable) $T_1 = T_2$ $G(j\omega)H(j\omega)$ locus passes through the $-1 + j0$ point $T_1 > T_2$ (Unstable)

Point	s	z
O	$0, +0j$	$\frac{K}{s^2}$ -> $\infty, -\infty$
A	$j\omega$	$\frac{K[1+T_2\omega j]}{-\omega^2 - T_1\omega^3 j} = \frac{K[-(\omega^2 + T_1T_2\omega^4) - \omega^3(T_2 - T_1)j]}{\omega^4 + (T_1\omega^3)^2}$
	$\omega \to \infty$	$-0 - 0j$ when $T_1 < T_2$ stable
		-0 when $T_1 = T_2$ unstable
		$-0 + 0j$ when $T_1 > T_2$ unstable

$$G(s)H(s) = \frac{K}{s(Ts - 1)}$$



Point	s	z
O	$0,+0j$	$-\frac{K}{s} \to -\infty, \infty j$
A	$j\omega$	$-\frac{K}{T\omega^2 + \omega j} = \frac{K[-T\omega^2 + \omega j]}{(T\omega^2)^2 + \omega^2}$
	$\omega \to \infty$	$-0 + 0j$ unstable

5.3.3 Bode Plots

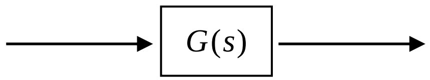
- Bode Plot of $G(s)$ is the pair of plots of the magnitude and phase of $G(j\omega)$ with the frequency ω serving as a parameter.

$$G(j\omega) = |G(j\omega)|e^{j\theta_G(\omega)} \tag{5.3.3-1}$$

where $|G(j\omega)|$ and $\theta_G(\omega)$ are known as the ***magnitude*** and ***phase*** functions of $G(s)$.

- Bode plot of the close-loop transfer function shows the frequency response at the steady state in form of magnitude and phase difference of the output compared to the input.

$$U(s)\frac{\omega}{s^2+\omega^2} \qquad Y(s) = \frac{G(s)\omega}{s^2+\omega^2} = H_1(s) + H_2(s) + \cdots + \frac{As+B}{s^2+\omega^2}$$



$$u(t) = \sin(\omega t) \qquad \underset{t\to\infty}{y(t)} = 0 + A\cos(\omega t) + \frac{B}{\omega}\sin(\omega t) = \sqrt{A^2 + (\frac{B}{\omega})^2}\sin(\omega t + \tan^{-1}(\frac{A\omega}{B}))$$

$$\underset{t\to\infty}{y(t)} = |G|\sin(\omega t + \angle(G))$$

Figure 5.3.3-1 Frequency Response

- Bode plot of the open-loop transfer function provides information of system stability, gain margin, and phase margin.

- Instead of plotting $|G(j\omega)|$ it is customary to plot

$$D(\omega) = 20\log_{10}|G(j\omega)| \tag{5.3.3-2}$$

- Regardless of the units of $G(s)$, the units of $D(\omega)$ are invariably decibels (dB). The plot of $D(\omega)$ versus ω is known as the Bode amplitude plot and the plot of $\theta_G(\omega)$ versus ω is known as the Bode phase plot.
- Consider a system having a loop transfer function

$$G(s) = G_0 \frac{\left(1+\frac{s}{z_1}\right)\cdots\left(1+\frac{s}{z_l}\right)}{\left(1+\frac{s}{p_1}\right)\cdots\left(1+\frac{s}{p_k}\right)} \tag{5.3.3-3}$$

$$G(j\omega) = G_0 \frac{\left(1+\frac{j\omega}{z_1}\right)\cdots\left(1+\frac{j\omega}{z_l}\right)}{\left(1+\frac{j\omega}{p_1}\right)\cdots\left(1+\frac{j\omega}{p_k}\right)} \tag{5.3.3-4}$$

$$|a+bj| = \sqrt{a^2+b^2} \tag{5.3.3-5}$$

$$\angle(a+bj) = \tan^{-1}\left(\frac{b}{a}\right) \tag{5.3.3-6}$$

$$\log\left(\frac{AB}{C}\right) = \log(A) + \log(B) - \log(C) \tag{5.3.3-7}$$

$$|G(j\omega)| = |G_0| \frac{\left(1+\left(\frac{\omega}{z_1}\right)^2\right)^{1/2} \cdots \left(1+\left(\frac{\omega}{z_l}\right)^2\right)^{1/2}}{\left(1+\left(\frac{\omega}{p_1}\right)^2\right)^{1/2} \cdots \left(1+\left(\frac{\omega}{p_k}\right)^2\right)^{1/2}} \quad (5.3.3\text{-}8)$$

$$D(\omega) = 20\log|G_0| + 10\log\left(1+\left(\frac{\omega}{z_1}\right)^2\right) + \cdots + 10\log\left(1+\left(\frac{\omega}{z_l}\right)^2\right)$$

$$-10\log\left(1+\left(\frac{\omega}{p_1}\right)^2\right) - \cdots - 10\log\left(1+\left(\frac{\omega}{p_k}\right)^2\right) \quad (5.3.3\text{-}9)$$

$$\theta_G(\omega) = \theta_0 + \tan^{-1}\left(\frac{\omega}{z_1}\right) + \cdots + \tan^{-1}\left(\frac{\omega}{z_l}\right) - \tan^{-1}\left(\frac{\omega}{p_1}\right) - \cdots - \tan^{-1}\left(\frac{\omega}{p_k}\right) \quad (5.3.3\text{-}10)$$

- When $\omega \to \infty$, $10\log\left(1+\left(\frac{\omega}{z}\right)^2\right) \to 20\log\left(\frac{\omega}{z}\right)$, the slope on log scale = 20 db, intersection point $\omega = z$
- When $\omega \to \infty$, $\tan^{-1}\left(\frac{\omega}{z}\right) \to 90°$
- If $G_0 > 0$ then $\theta_0 = 0$, but if $G_0 < 0$ then $\theta_0 = 180°$.
- At each zero or pole the log-magnitude plot increases or decreases its magnitude with the asymptotic-slope of 20log(ω)/decade with the real magnitude of 10log(2) = 3.010 dB at that pint.

- The phase plot increases or decreases with the asymptotic-line of 90° with the real phase of 45° at that point.

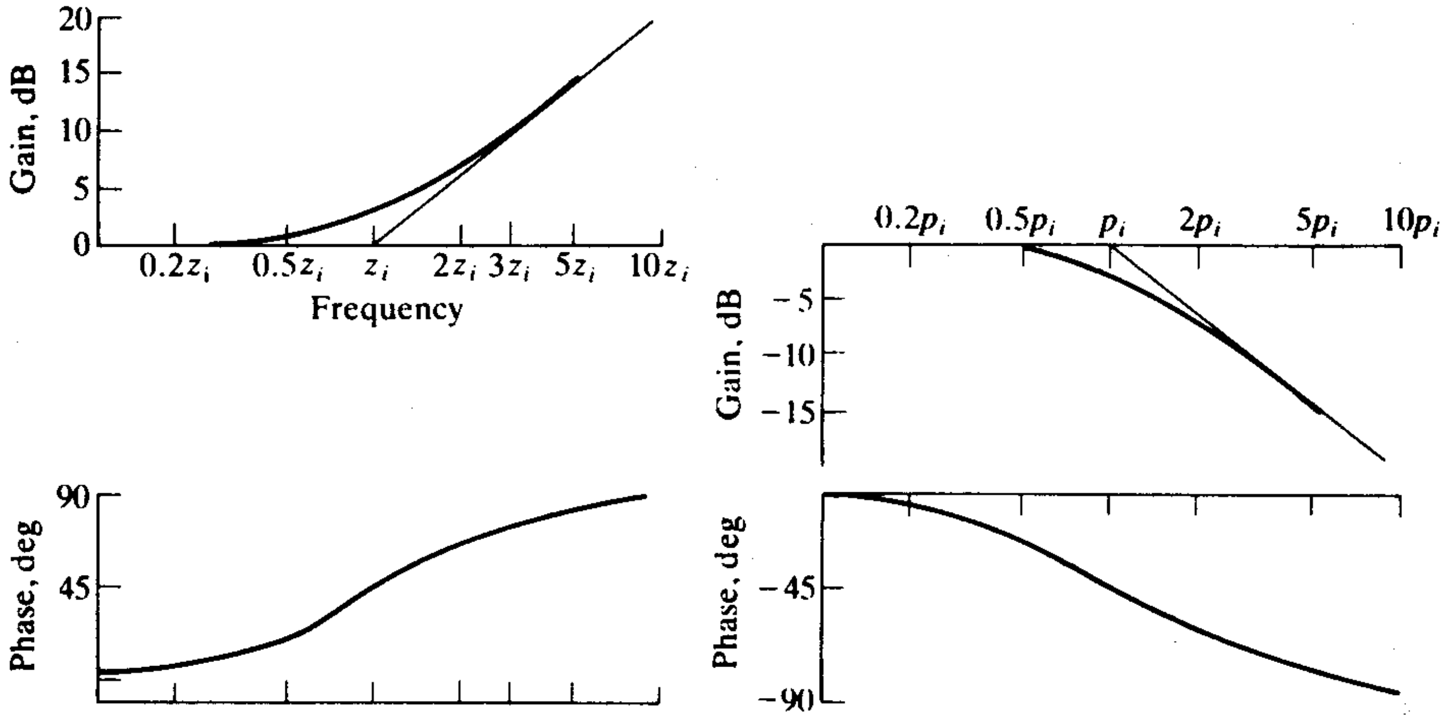


Figure 5.3.3-1 Bode Plots for a Zero and for a Pole (a) $G(s) = 1 + \frac{s}{z_i}$ (b) $G(s) = \frac{1}{1 + \frac{s}{p_i}}$

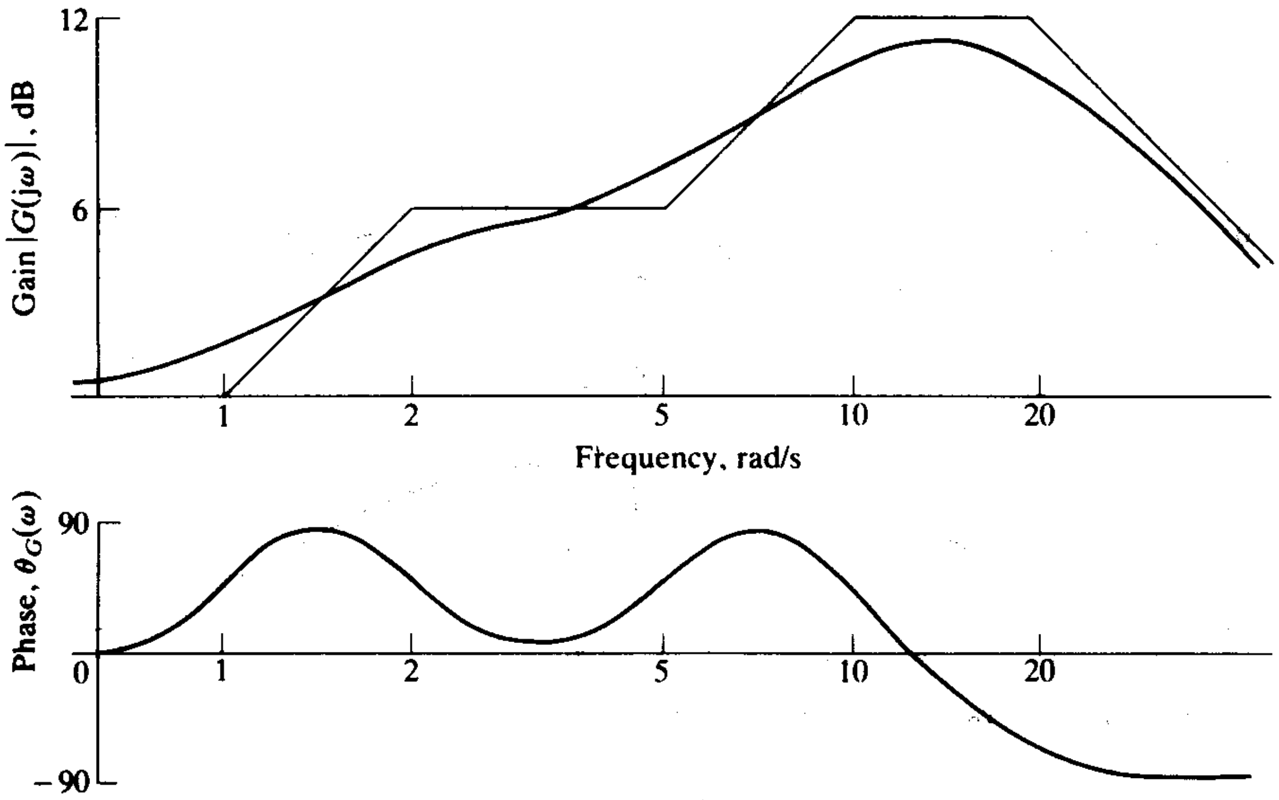


Figure 5.3.3-2 Bode Plots for $G(s) = \dfrac{(1+s)(1+\frac{s}{5})}{(1+\frac{s}{2})(1+\frac{s}{10})(1+\frac{s}{20})}$

- Zero or pole at the origin increases or decreases the log-magnitude with a constant slope of 20 dB/decade decade with the real magnitude of -∞ or ∞ at the origin, where as the phase plot increases or decreases with a constant phase of 90°.

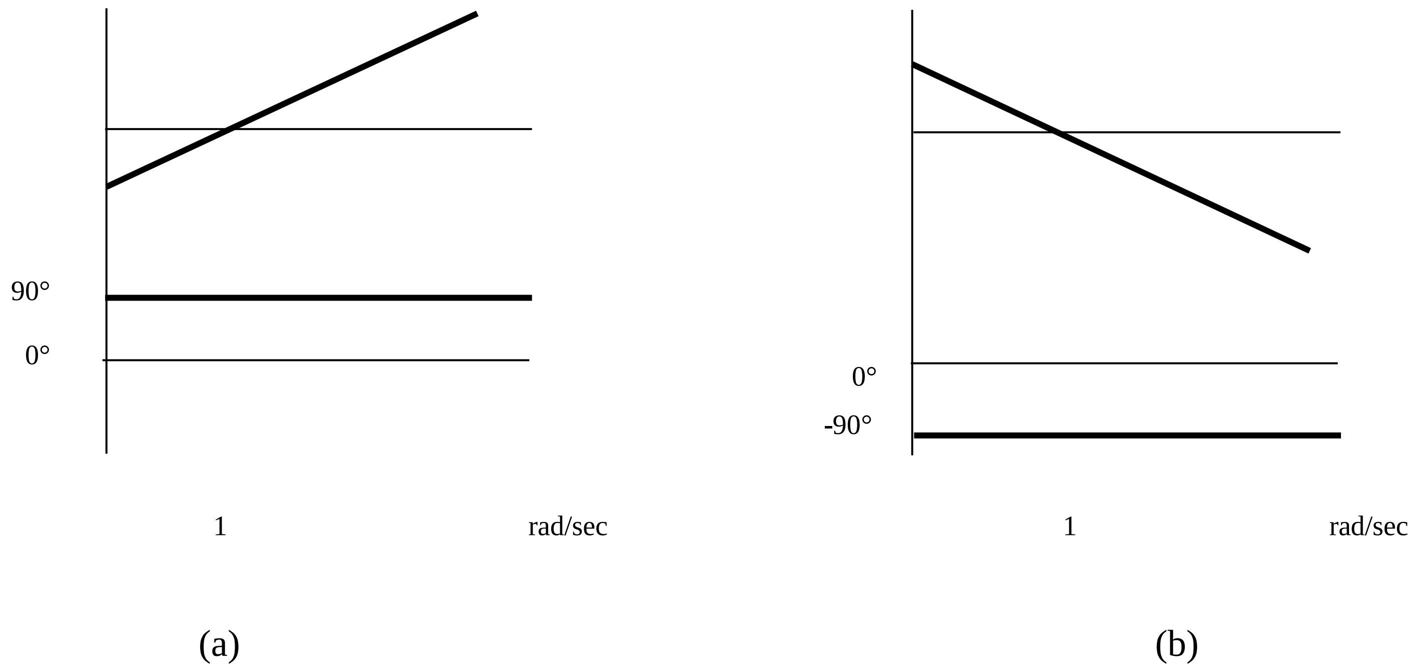


Figure 5.3.3-3 Bode Plots for (a) $G(s) = s$ (b) $G(s) = \frac{1}{s}$

For second order system,

$$G_i(s) = \frac{1}{1 + 2\zeta(s/\omega_0) + (s/\omega_0)^2} \tag{5.3.3-11}$$

$$G_i(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_0^2}\right) + \frac{2\zeta\omega}{\omega_0} j} \tag{5.3.3-12}$$

$$|G_i(s)| = -10\log\left[\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_0}\right)^2\right] \tag{5.3.3-13}$$

$$\angle G_i(s) = -\tan^{-1}\frac{2\zeta\omega/\omega_0}{1 - (\omega/\omega_0)^2} \tag{5.3.3-14}$$

- When $\omega \to \infty$, $-10\log\left[\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_0}\right)^2\right] \to -10\log\left(\frac{\omega}{z}\right)^4$, the slope on log scale = -40 db, intersection point $\omega = \omega_0$
- When $\omega \to \infty$, $-\tan^{-1}\frac{2\zeta\omega/\omega_0}{1 - (\omega/\omega_0)^2} \to -180°$

$$\omega_r = \sqrt{1 - 2\zeta^2}\,\omega_0 \tag{5.3.3-15}$$

- There is no resonance peak for $\zeta > 1/\sqrt{2}$.

$$D_i(\omega_r) = -20\log[4\varsigma^2(1 - \varsigma^2)] \text{ dB} \tag{5.3.3-16}$$

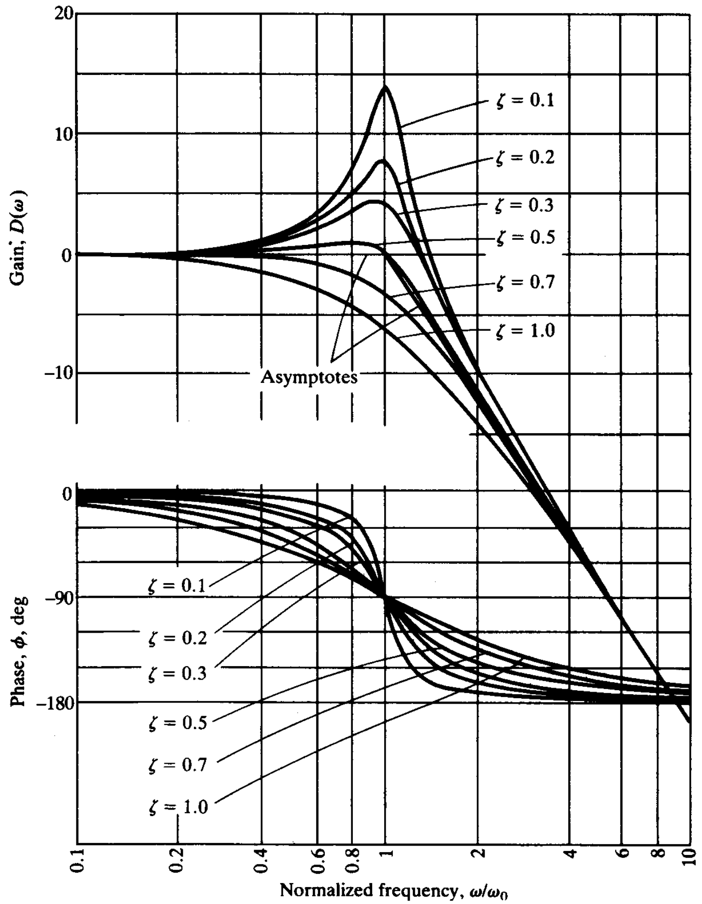


Figure 5.3.3-4 Bode Plot for $G_i(s) = \dfrac{1}{1 + 2\zeta(s/\omega_0) + (s/\omega_0)^2}$

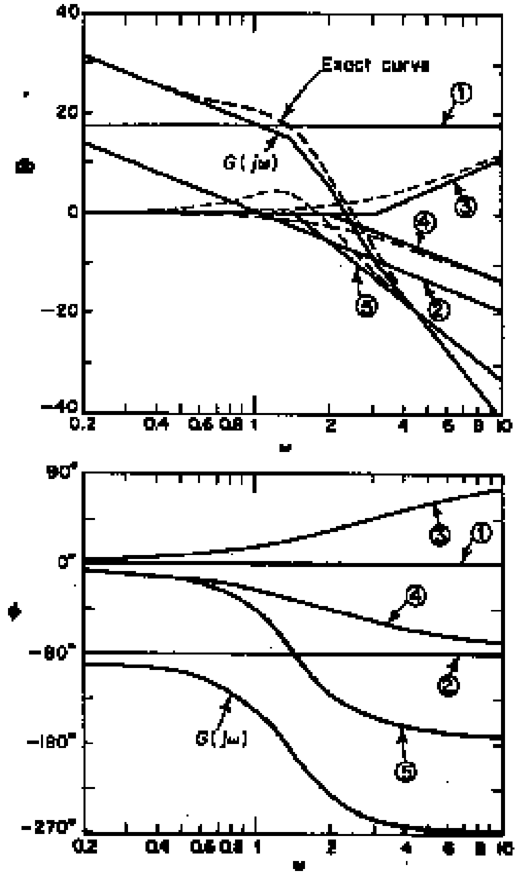


Figure 5.3.3-5 Bode Plot for $G(s)=\dfrac{10(s+3)}{s(s+2)(s^2+s+2)}=\dfrac{7.5\left(\frac{s}{3}+1\right)}{s\left(\frac{s}{2}+1\right)\left[\left(\frac{s}{\sqrt{2}}\right)^2+\frac{s}{2}+1\right]}$

5.4 Steady-State Response

- A system designed to follow a reference input rather than merely to return to equilibrium is generally known as a ***tracking system***.

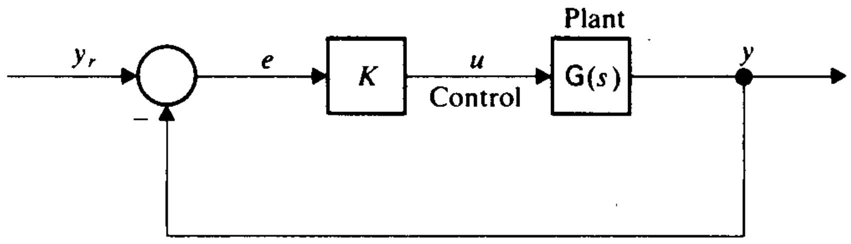


Figure 5.4-1 Error Driven Feedback Control System

$$e = y_r - y \tag{5.4-1}$$

$$H_E(s) = \frac{1}{1+KG(s)} = \frac{e(s)}{y_r(s)} \tag{5.4-2}$$

$$H_C(s) = \frac{KG(s)}{1+KG(s)} = \frac{y(s)}{y_r(s)} \tag{5.4-3}$$

Definition: A system is of type *m* if it can track a polynomial input of degree *m* with finite but nonzero steady state error.

From final-value theorem

$$\text{Steady State Error} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) \tag{5.4-4}$$

$$e(s) = \frac{1}{1+KG(s)} y_r(s) \tag{5.4-5}$$

When the input $y_r(t)$ is the polynomial function,

$$y_r(t) = p_m(t) = C_1 + \frac{C_2}{1} t + \cdots + \frac{C_{m+1}}{m!} t^m \tag{5.4-6}$$

$$y_r(s) = p_m(s) = \frac{C_1}{s} + \frac{C_2}{s^2} + \cdots + \frac{C_{m+1}}{s^{m+1}} \text{ and } C_{m+1} \neq 0$$

$$= \frac{C_1 s^m + C_2 s^{m-1} + \cdots + C_{m+1}}{s^{m+1}} \tag{5.4-7}$$

$$se(s) = \frac{1}{1+KG(s)} \frac{C_1 s^m + C_2 s^{m-1} + \cdots + C_{m+1}}{s^m} \tag{5.4-8}$$

The limit as $s \to 0$ of $se(s)$ is finite only if $G(s)$ has pole of the proper order at $s = 0$.

$$G(s) = \frac{N(s)}{s^p D'(s)} \tag{5.4-9}$$

$$se(s) = \frac{s^p D'(s)}{s^p D'(s) + KN(s)} \frac{C_1 s^m + C_2 s^{m-1} + \cdots + C_{m+1}}{s^m} \tag{5.4-10}$$

if $p > m$, $\lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) = 0$

if $p = m$, $\lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) =$ is finite but nonzero (5.4-11)

if $p < m$, $\lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) =$ is infinite

- The system type is equal to the order of the pole of $G(s)$ at $s = 0$.
- Since a pole at the origin represents a perfect integrator, the system type is often defined as the number of cascaded integrators in the system.
- The steady-state error that results when the polynomial input is the same degree as the system type is determined to be

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} se(s) = \begin{cases} \dfrac{C_1 D(0)}{D(0)+KN(0)}, & p = m = 0 \\ \dfrac{C_{m+1} D(0)}{KN(0)}, & p = m \geq 0 \end{cases} \qquad (5.4\text{-}12)$$

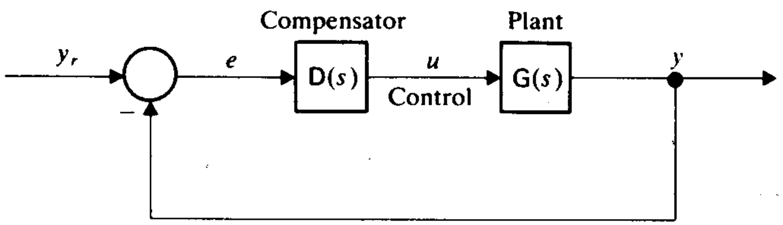
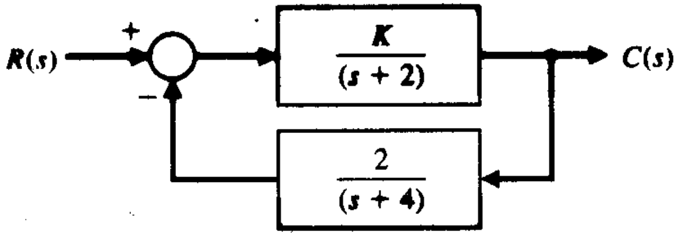


Figure 5.4-2 Error Driven Feedback Control System with Compensator

The most common type of compensator used to improve the tracking performance is the PI (proportional + integral) compensator.

$$D(s) = \frac{K_i}{s} + K_p = \frac{K_i + K_p s}{s} \qquad (5.4\text{-}13)$$

Let's consider the system shown in figure below



$$\frac{E(s)}{R(s)} = \frac{s^2 + 6s + 8}{s^2 + 6s + 8 + 2K} \quad (1)$$

By final-value theorem

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{s^2 + 6s + 8}{s^2 + 6s + 8 + 2K} R(s) \quad (2)$$

For step input, $R(s) = \frac{1}{s}$,

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{s^2 + 6s + 8}{s^2 + 6s + 8 + 2K} \frac{1}{s} = \frac{8}{8 + 2K} \quad (3)$$

5.5 Dynamic Response

The dynamic characteristics of the system are typically defined in terms of the response to a ***unit-step input***.

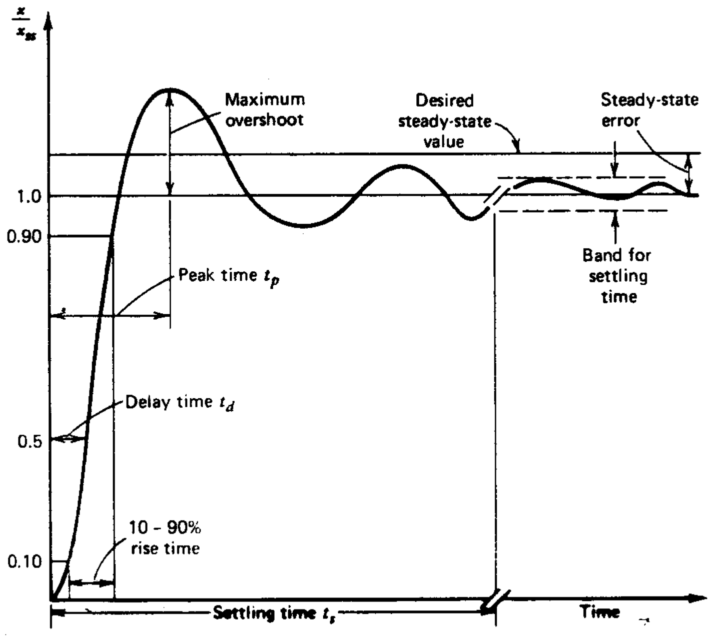


Figure 5.5-1 Characteristics of Dynamic Response

1. **Characteristic roots:** $s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m} = s_1, s_2$

2. Stability: Stable if m, c, and k have the same sign.

3. Damping ratio or damping factor: $\zeta = \dfrac{c}{2\sqrt{mk}}$

4. Undamped natural frequency: $\omega_n = \sqrt{\dfrac{k}{m}}$

5. Damped natural frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

6. Time constant: $\tau = 2m/c = 1/\zeta\omega_n$ if $\zeta \leq 1$.
   $\tau = -1/s_1$ if $\zeta > 1$ (s_1 = dominant root)

Step Response Characteristics

7. Maximum percent overshoot: $100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

8. Peak time: $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

9. Delay time: $t_d \simeq \dfrac{1+0.7\zeta}{\omega_n}$, $0 \leq \zeta \leq 1$

10. 100% rise time: $t_r|_{100\%} = \dfrac{2\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$ $\tan\phi = \dfrac{\sqrt{1-\zeta^2}}{\zeta}$ (third quadrant)

11. **2% settling** time: $t_s = 4\tau$ for $\zeta < 0.7$.

Table 5.5-1 Useful Formulas for Second-order Model: $m\ddot{x} + c\dot{x} + kx = f$

- ***Response time*** is defined as the centroid $\bar{t}$ of the impulse response of the system.

$$\bar{t} = \frac{\int_0^\infty t h(t)dt}{\int_0^\infty h(t)dt} \quad (5.5\text{-}1)$$

$$H(s) = \int_0^\infty e^{-st} h(t)dt \quad (5.5\text{-}2)$$

$$H(0) = \int_0^\infty h(t)dt \quad (5.5\text{-}3)$$

$$H'(s) = dH(s)/ds = -\int_0^\infty t e^{-st} h(t)dt \quad (5.5\text{-}4)$$

$$-H'(0) = \int_0^\infty t h(t)dt \quad (5.5\text{-}5)$$

$$\bar{t} = -\frac{H'(0)}{H(0)} \quad (5.5\text{-}6)$$

- Response time of first-order system

$$H(s) = \frac{\omega_0}{s + \omega_0} \tag{5.5-7}$$

$$H'(s) = -\frac{\omega_0}{(s + \omega_0)^2} \tag{5.5-8}$$

$$\bar{t} = -\frac{H'(0)}{H(0)} = \frac{1}{\omega_0} \tag{5.5-9}$$

The step response corresponding to this transfer function is

$$Y(s) = \frac{\omega_0}{s + \omega_0} \cdot \frac{1}{s} = \frac{1}{s} - \frac{1}{s + \omega_0} \tag{5.5-10}$$

$$y(t) = L^{-1}\left[\frac{\omega_0}{s(s + \omega_0)}\right] = 1 - e^{-\omega_0 t} \tag{5.5-11}$$

The step response reaches 1-e^{-1} of its final value at $t_{0.63}$ given by

$$\omega_0 t_{0.63} = 1 \tag{5.5-12}$$

For a first-order system, the centroidal response time and the 63 percent response time are exactly equal.

- Response time of second-order system

$$H(s) = \frac{\omega_0{}^2}{s^2 + 2\zeta\omega_0 s + \omega_0{}^2} \tag{5.5-13}$$

$$H'(s) = -\frac{\omega_0{}^2(2s + s\zeta\omega_0)}{(s^2 + 2\zeta\omega_0 s + \omega_0{}^2)^2} \tag{5.5-14}$$

$$\bar{t} = -\frac{H'(0)}{H(0)} = \frac{2\zeta}{\omega_0} \tag{5.5-15}$$

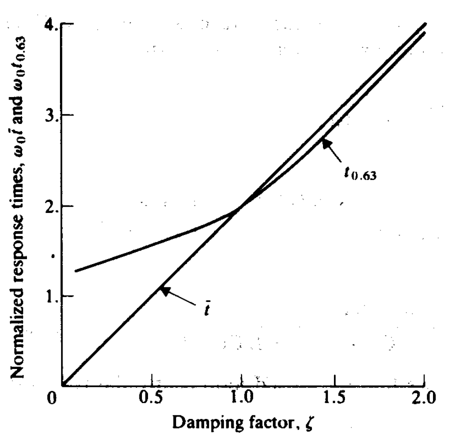


Figure 5.5-2 Response Time of Second-Order System $H(s) = \frac{\omega_0{}^2}{s^2 + 2\zeta\omega_0 s + \omega_0{}^2}$

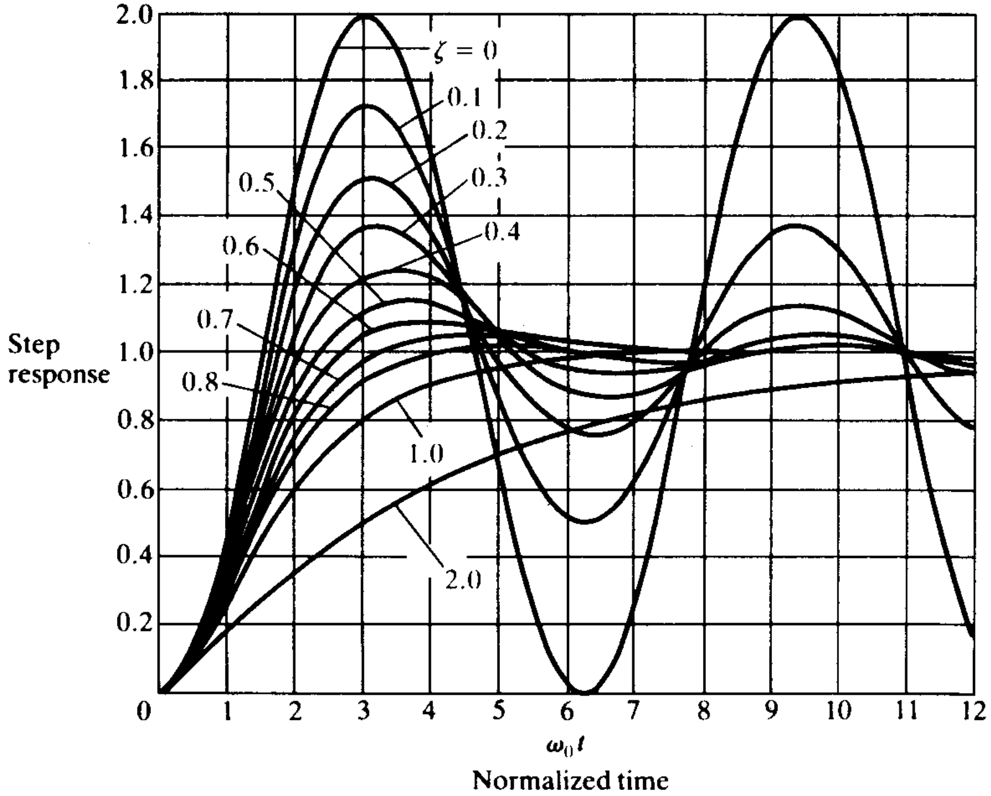


Figure 5.5-3 Step Response of Second-Order System $H(s) = \dfrac{\omega_0^{\,2}}{s^2 + 2\zeta\omega_0 s + \omega_0^{\,2}}$

- Rise time of systems in tandem, suppose a system $H(s)$ comprises two subsystems $H_1(s)$ and $H_2(s)$ in tandem.

$$H(s) = H_1(s)H_2(s) \tag{5.5-16}$$

$$H'(s) = H_1'(s)H_2(s) + H_1(s)H_2'(s) \tag{5.5-17}$$

$$\frac{H'(s)}{H(s)} = \frac{H_1'(s)H_2(s) + H_1(s)H_2'(s)}{H_1(s)H_2(s)} = \frac{H_1'(s)}{H_1(s)} + \frac{H_2'(s)}{H_2(s)} \tag{5.5-18}$$

$$\bar{t} = \bar{t}_1 + \bar{t}_2 \tag{5.5-19}$$

- Rise time of feedback system, suppose $H(s)$ is the transfer function of a closed-loop control system.

$$H(s) = \frac{KG(s)}{1 + KG(s)} \tag{5.5-20}$$

$$H'(s) = \frac{[1 + KG(s)]KG'(s) - KG(s)KG'(s)}{[1 + KG(s)]^2} = \frac{KG'(s)}{[1 + KG(s)]^2} \tag{5.5-21}$$

$$\frac{H'(s)}{H(s)} = \frac{KG'(s)}{[1 + KG(s)]^2}\frac{1 + KG(s)}{KG(s)} = \frac{1}{1 + KG(s)}\frac{G'(s)}{G(s)} \tag{5.5-22}$$

$$\bar{t} = \frac{1}{1 + KG(0)}\bar{t}_G \tag{5.5-23}$$

5.6 Robustness and Stability

- In reality, no mathematical model can predict the behavior of a physical process exactly. Some uncertainty of the physical process always exists.
- Gain margin is the amount that the loop gain can be changed, at the frequency at which the phase shift is 180°, without reducing the return difference to zero.
- Phase margin is the amount of phase lag that can be added to the open-loop transfer function, at the frequency at which its magnitude is unity, without making the return difference zero.

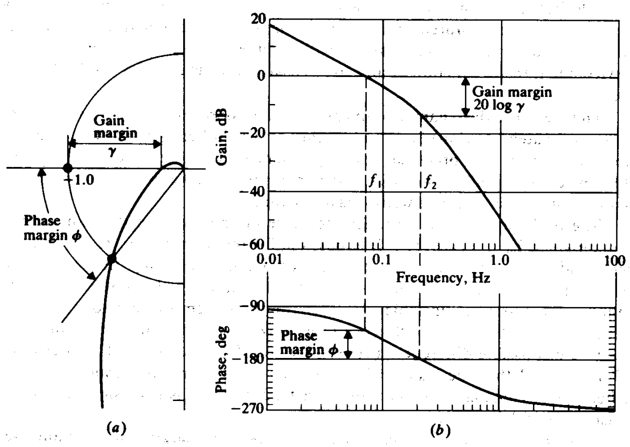


Figure 5.6-1 Gain and Phase Margins Defined (a) Nyquist Diagram (b) Bode Plots

6 Control Laws

- The algorithm that is physically designed to act on the actuating (error) signal to produce the control signal is ***the control law*** or ***control action***.
- The control objectives might be stated as follows:

1. Minimize the steady-state error.
2. Minimize the response time.
3. Achieve other transient specifications, such as minimizing the maximum overshoot.

6.1 Two-Position Control

- With the ***on-off controller***, the controller output is either on or off. The controller output is determined by the magnitude of the error signal.

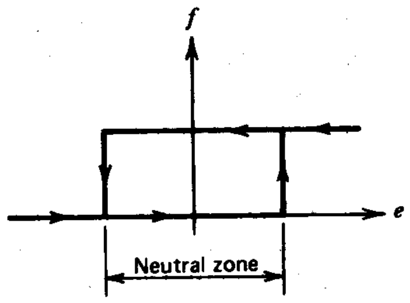


Figure 6.1-1 Transfer Characteristics of the On-Off Controller

- The actuating error is $e = r - c$, where r is set point, c is controlled variable, and f is control signal.
- The controlled variable cycles with an amplitude that depends on the width of the neutral zone. This zone is provided to prevent frequent on-off switching, or chattering, which can shorten the life of the device.
- The cycling frequency also depends on the time constant of the controlled process and the magnitude of the control signal.
- Application examples: heater, air conditioner

- ***Bang-bang controller*** is distinguished from on-off control by the fact that the direction or sign of the control signal can have two values.

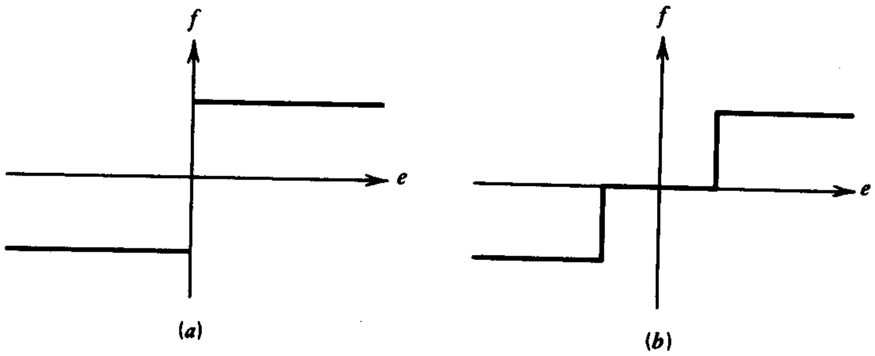


Figure 6.1-2 Transfer Characteristics of the Bang-Bang Controller

(a) Ideal Bang-Bang Control (b) Bang-Bang Control With a Dead Zone

- Application examples: pneumatic system, toy motors

6.2 Proportional Control

- Proportional control is the algorithm in which the change in the control signal is proportional to the error.

$$F(s) = K_p E(s) \tag{6.2-1}$$

where $F(s)$ is the deviation in the control signal and K_p is the proportional gain.

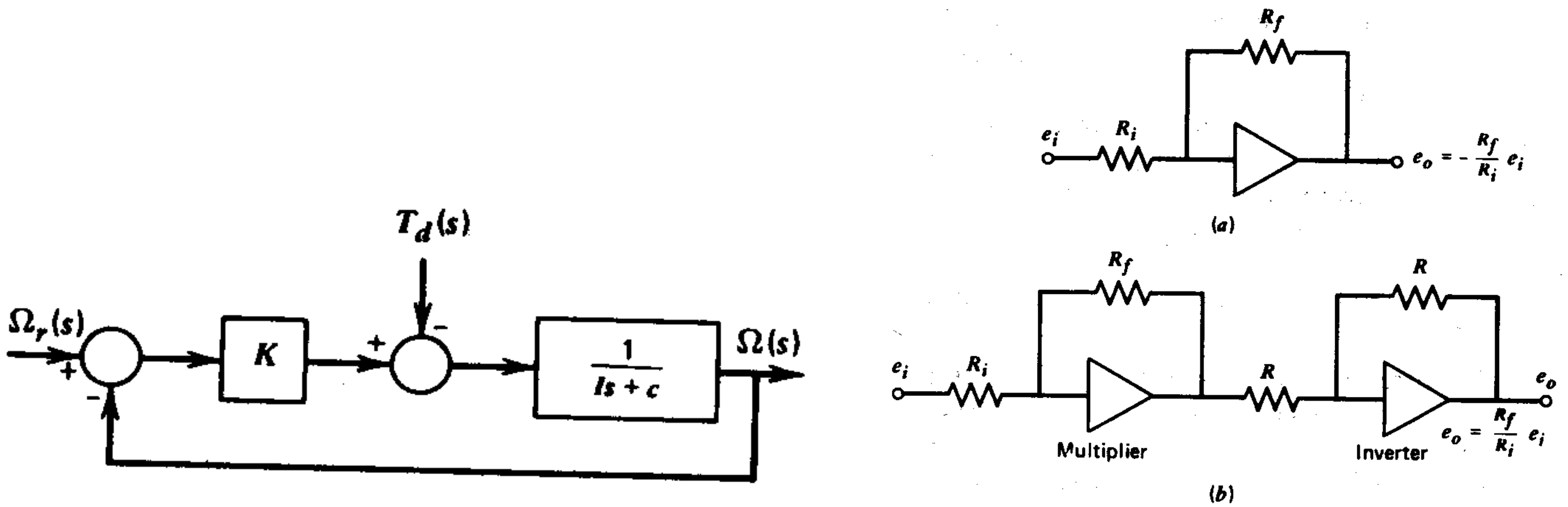


Figure 6.2-1 Simplified Form of Speed Control System Using Proportional Control

Proportional Control of a First-Order System

$$\frac{\Omega(s)}{\Omega_r(s)} = \frac{K}{Is + c + K} \tag{6.2-2}$$

$$\frac{\Omega(s)}{T_d(s)} = \frac{-1}{Is + c + K} \tag{6.2-3}$$

- From Routh-Hurwitz stability criteria, the system is stable when $I/(c+K)$ is positive.

		I
	α	$c+K$
1	$I/(c+K)$	0

- Steady-state output responding to unit-step command

$$\Omega(s) = \frac{K}{Is + c + K} \cdot \frac{1}{s} \tag{6.2-4}$$

$$\omega_{ss} = \lim_{s \to 0} s \frac{K}{Is + c + K} \frac{1}{s} = \frac{K}{c + K} < 1 \tag{6.2-5}$$

- Steady-state error responding to unit-step disturbance

$$\Omega(s) = \frac{-1}{Is + c + K} \frac{1}{s} \tag{6.2-6}$$

$$\omega_{ss} = \lim_{s \to 0} s \frac{-1}{Is + c + K} \frac{1}{s} = -\frac{1}{c + K} \tag{6.2-7}$$

- Time constant

$$\tau = \frac{I}{c + K} \tag{6.2-8}$$

- For a first-order system whose inputs are step functions,

1. The output never reaches its desired value even in the absence of a disturbance if damping is present ($c \neq 0$), although it can be made arbitrarily close by choosing the gain K large enough. This is ***offset error***.
2. The output approaches its final value without oscillation. The time to reach this value is inversely proportional to K.
3. The output error due to the disturbance at steady state is inversely proportional to the gain K. This error is present even in the absence of damping.

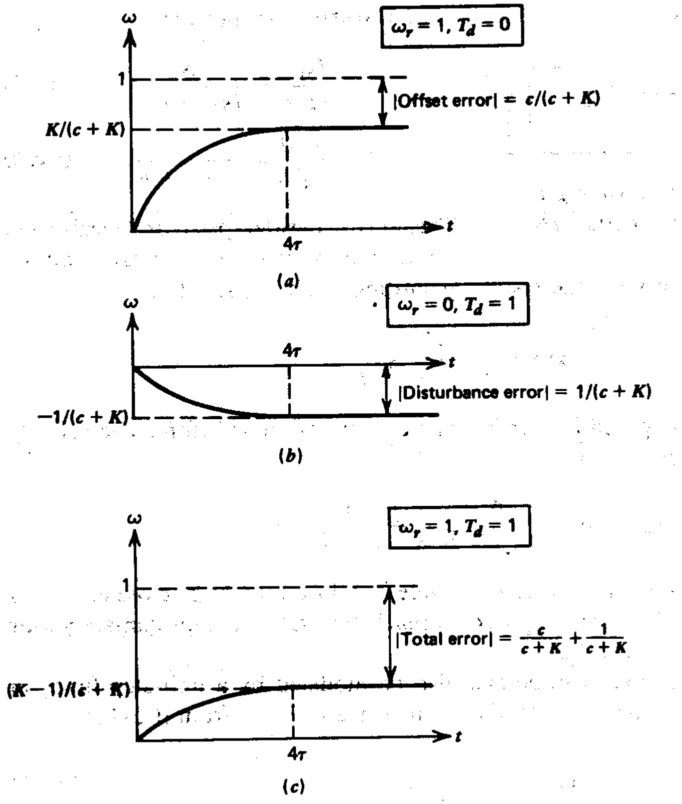


Figure 6.2-2 Comparison of Two Sources of Error in a First-Order System

(a) System Response to a Unit Step Reference Input, with No Disturbance

(b) System Response to a Unit Step Disturbance with Zero Reference Input

(c) Total Response to Both the Unit Step of Reference Input and the Disturbance

Feedforward Compensation

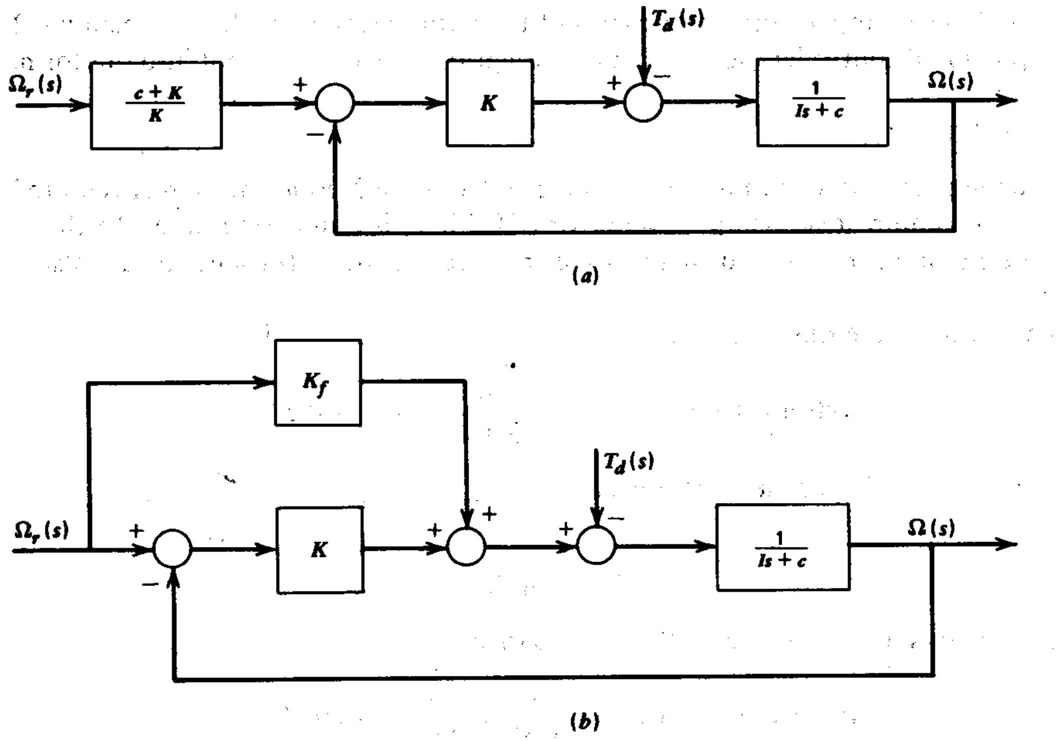


Figure 6.2-3 Input Compensation to augment Proportional Control. (a) This arrangement eliminates error with a step input. (b) Feedforward compensation eliminates the errors due to both step and ramp inputs.

- To make a zero steady-state offset error

1. By giving a command input that is $(c+K)/K$ times higher than the desired speed, ***input compensation***.
2. By feedforward compensation, the primary transfer function is

$$\frac{\Omega(s)}{\Omega_r(s)} = \frac{K_f + K}{Is + c + K} \tag{6.2-9}$$

- For a unit-step input, $\omega_{ss} = (K_f + K) / (c + K)$, if $K_f = c$, then $\omega_{ss} = 1$, and the offset error is zero.
- Note that the feedforward compensation does not affect the disturbance response.

Proportional Control of a Second-Order System

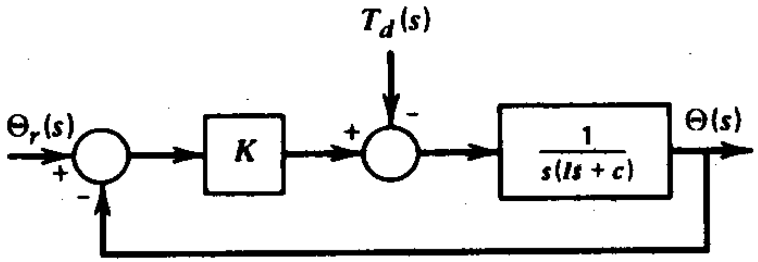


Figure 6.2-4 Simplified Form of Position Control System Using Proportional Control

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K}{Is^2 + cs + K} \quad (6.2\text{-}10)$$

$$\frac{\Theta(s)}{T_d(s)} = \frac{-1}{Is^2 + cs + K} \quad (6.2\text{-}11)$$

- From Routh-Hurwitz stability criteria, the system is stable when I/c and c/K are positive.

		I	K
	α	c	0
1	I/c	K	0
2	c/K	0	0

- Steady-state output responding to unit-step command

$$\theta_{ss} = \frac{K}{K} = 1 \tag{6.2-12}$$

- Steady-state error responding to unit-step disturbance

$$\theta_{ss} = \frac{-1}{K} \tag{6.2-13}$$

$$Is^2 + cs + K = 0 \rightarrow s^2 + \frac{c}{I}s + \frac{K}{I} = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{6.2-14}$$

- Natural frequency

$$\omega_n = \sqrt{\frac{K}{I}} \tag{6.2-15}$$

- Damping ratio

$$\zeta = \frac{c}{2\sqrt{IK}} \tag{6.2-16}$$

- Time constant

$$\tau = \frac{2I}{c} \tag{6.2-17}$$

Let's consider a general second-order system represented by its transfer function

$$G_p = \frac{1}{Is^2 + cs + k} \tag{1}$$

The closed loop transfer functions

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K}{Is^2 + cs + (k + K)} \tag{2}$$

$$\frac{\Theta(s)}{T_d(s)} = \frac{-1}{Is^2 + cs + (k + K)} \tag{3}$$

From Routh-Hurwitz stability criteria, the system stable is stable when I/c and $c/(k+K)$ are positive.

		I	$k+K$
	α	c	0
1	I/c	$k+K$	0
2	$c/(k+K)$	0	0

Steady-state output responding to unit-step command

$$\theta_{ss} = \frac{K}{k + K} \tag{4}$$

Steady-state error responding to unit-step disturbance

$$\theta_{ss} = -\frac{1}{k + K} \tag{5}$$

$$Is^2 + cs + (k + K) = 0 \rightarrow s^2 + \frac{c}{I}s + \frac{(k + K)}{I} = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad (6)$$

Damping ratio

$$\zeta = \frac{c}{2\sqrt{I(k + K)}} \qquad (8)$$

Time constant

$$\tau = \frac{2I}{c} \qquad (7)$$

6.3 Integral Control

- Integral control improves the offset error that occurs with proportional control.

$$F(s) = \frac{K_I}{s} E(s) \tag{6.3-1}$$

$$f(t) = K_I \int_0^t e(t)dt \tag{6.3-2}$$

where $F(s)$ is the deviation in the control signal and K_I is the integral gain.

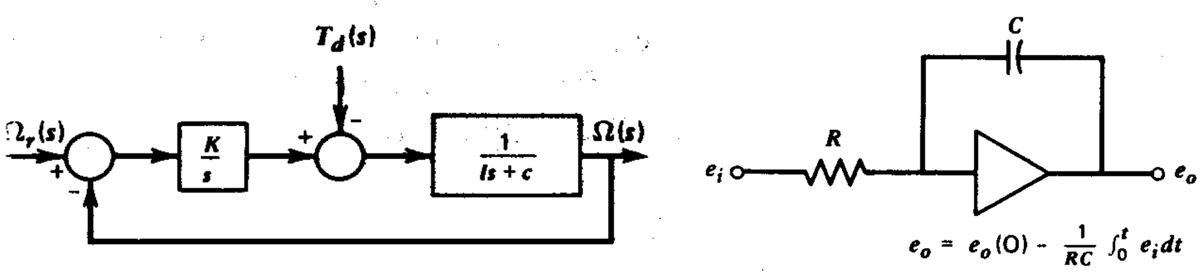


Figure 6.3-1 Simplified Form of Velocity Control System Using Integral Control

Integral Control of a First-Order System

$$\frac{\Omega(s)}{\Omega_r(s)} = \frac{K}{Is^2 + cs + K} \tag{6.3-3}$$

$$\frac{\Omega(s)}{T_d(s)} = \frac{-s}{Is^2 + cs + K} \tag{6.3-4}$$

- From Routh-Hurwitz stability criteria, the system is stable when *I*/*c* and *c*/*K* are positive.

		I	K
	α	c	0
1	I/c	K	0
2	c/K	0	0

- Steady-state output responding to unit-step command

$$\omega_{ss} = \frac{K}{K} = 1 \tag{6.3-5}$$

- Steady-state error responding to unit-step disturbance

$$\omega_{ss} = 0 \tag{6.3-6}$$

$$Is^2 + cs + K = 0 \rightarrow s^2 + \frac{c}{I}s + \frac{K}{I} = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad (6.3\text{-}7)$$

- Damping ratio

$$\zeta = \frac{c}{2\sqrt{IK}} \qquad (6.3\text{-}8)$$

- Time constant

$$\tau = \frac{2I}{c} \qquad (6.3\text{-}9)$$

Integral Control of a Second-Order System

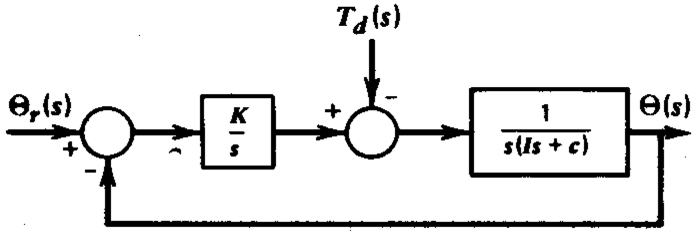


Figure 6.3-2 Simplified Form of Position Control System Using Integral Control

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K}{Is^3 + cs^2 + K} \qquad (6.3\text{-}11)$$

$$\frac{\Theta(s)}{T_d(s)} = \frac{-s}{Is^3 + cs^2 + K} \qquad (6.3\text{-}12)$$

- With the Routh criterion, we immediately see that the system is not stable because of the missing *s* term.

Let's consider a general second-order system represented by its transfer function

$$G_p = \frac{1}{Is^2 + cs + k} \tag{1}$$

The closed loop transfer functions

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K}{Is^3 + cs^2 + ks + K} \tag{2}$$

$$\frac{\Theta(s)}{T_d(s)} = \frac{-s}{Is^3 + cs^2 + ks + K} \tag{3}$$

From Routh-Hurwitz stability criteria, the system stable is stable when I/c, $c^2/(kc\text{-}KI)$, and $(kc\text{-}KI)/(cK)$ are positive.

		I	k
	α	c	K
1	I/c	$k\text{-}KI/c$	0
2	$c^2/(kc\text{-}KI)$	K	0
3	$(kc\text{-}KI)/(cK)$	0	0

Steady-state output responding to unit-step command

$$\theta_{ss} = 1 \tag{4}$$

Steady-state error responding to unit-step disturbance

$$\theta_{ss} = 0 \tag{5}$$

6.4 Proportional-Plus-Integral Control

- Proportional-plus-integral control is used to improve both steady-state and transient response of the system.

$$F(s) = K_P E(s) + \frac{K_I}{s} E(s) \tag{6.4-1}$$

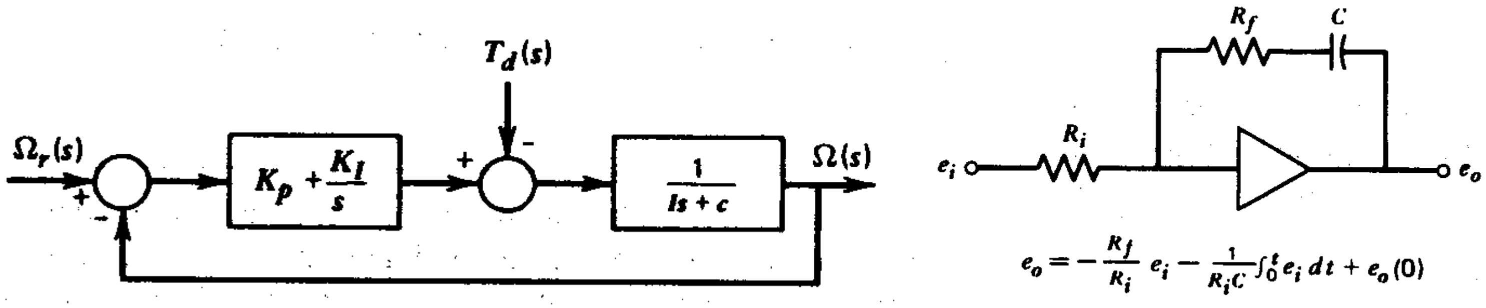


Figure 6.4-1 Simplified Form of Velocity Control System Using PI Control

PI Control of a First-Order System

$$\frac{\Omega(s)}{\Omega_r(s)} = \frac{K_p s + K_I}{Is^2 + (c + K_p)s + K_I} \tag{6.4-2}$$

$$\frac{\Omega(s)}{T_d(s)} = \frac{-s}{Is^2 + (c + K_p)s + K_I} \tag{6.4-3}$$

- From Routh-Hurwitz stability criteria, the system is stable when $I/(c+K_p)$ and $(c+K_p)/K_I$ are positive.

		I	K_I
	α	$c+K_p$	0
1	$I/(c+K_p)$	K_I	0
2	$(c+K_p)/K_I$	0	0

- Steady-state output responding to unit-step command

$$\omega_{ss} = \frac{K_I}{K_I} = 1 \tag{6.4-4}$$

- Steady-state error responding to unit-step disturbance

$$\omega_{ss} = 0 \tag{6.4-5}$$

$$Is^2 + (c + K_p)s + K_I = 0 \rightarrow s^2 + \frac{(c + K_p)}{I}s + \frac{K_I}{I} = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{6.4-6}$$

- Damping ratio

$$\zeta = \frac{c + K_p}{2\sqrt{IK_I}} \tag{6.4-7}$$

- Time constant

$$\tau = \frac{2I}{c + K_p} \tag{6.4-8}$$

PI Control of a Second-Order System

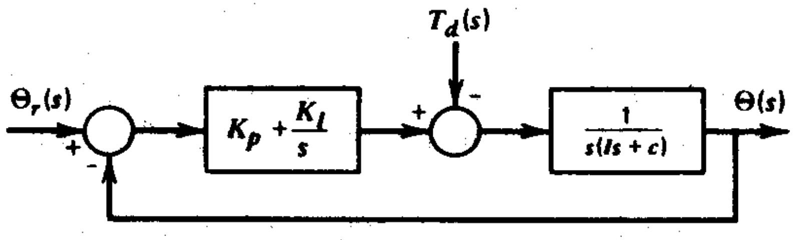


Figure 6.4-2 Simplified Form of Position Control System Using PI Control

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K_p s + K_I}{Is^3 + cs^2 + K_p s + K_I} \tag{6.4-9}$$

$$\frac{\Theta(s)}{T_d(s)} = \frac{-s}{Is^3 + cs^2 + K_p s + K_I} \tag{6.4-10}$$

- From Routh-Hurwitz stability criteria, the system stable is stable when I/c, $c^2/(K_pc-K_II)$, and $(K_pc-KI)/(cK)$ are positive.

		I	K_p
	α	c	K_I
1	I/c	K_p-K_II/c	0
2	$c^2/(K_pc-K_II)$	K_I	0
3	$(K_pc-K_II)/(cK_I)$	0	0

- Steady-state output responding to unit-step command

$$\theta_{ss} = 1 \qquad (6.4\text{-}11)$$

- Steady-state error responding to unit-step disturbance

$$\theta_{ss} = 0 \qquad (6.4\text{-}12)$$

Let's consider a general second-order system represented by its transfer function

$$G_p = \frac{1}{Is^2 + cs + k} \quad (1)$$

The closed loop transfer functions

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K_p s + K_I}{Is^3 + cs^2 + (k + K_p)s + K_I} \quad (2)$$

$$\frac{\Theta(s)}{T_d(s)} = \frac{-s}{Is^3 + cs^2 + (k + K_p)s + K_I} \quad (3)$$

From Routh-Hurwitz, the system stable is stable when I/c, $c^2/(kc+K_pc-K_II)$, and $(kc+K_pc-K_II)/(cK_I)$ are positive.

		I	$k+K_p$
	α	c	K_I
1	I/c	$k+K_p-K_II/c$	0
2	$c^2/(kc+K_pc-K_II)$	K_I	0
3	$(kc+K_pc-K_II)/(cK_I)$	0	0

Steady-state output responding to unit-step command

$$\theta_{ss} = 1 \quad (4)$$

Steady-state error responding to unit-step disturbance

$$\theta_{ss} = 0 \quad (5)$$

6.5 Derivative Control

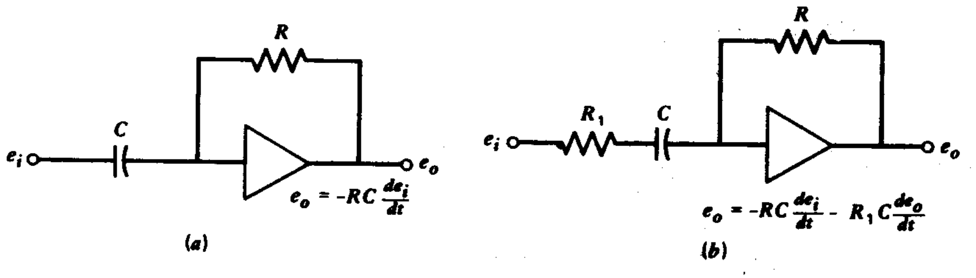


Figure 6.5-1 PD Control

- Derivative control is used to damp out oscillations.

$$F(s) = K_D sE(s) \tag{6.5-1}$$

$$f(t) = K_D \frac{d}{dt} e(t) \tag{6.5-2}$$

where K_D is the derivative gain.

- Since it depends on only the error rate, derivative control should never be used alone.
- PD and PID control algorithms

$$F(s) = (K_p + K_D s)E(s) \tag{6.5-3}$$

$$F(s) = (K_p + \frac{K_I}{s} + K_D s)E(s) \tag{6.5-4}$$

Let's consider a first-order system represented by its transfer function

$$G_p = \frac{1}{Is + c} \quad (1)$$

The closed loop transfer functions

$$\frac{\Omega(s)}{\Omega_r(s)} = \frac{K_D s + K_p}{(I + K_D)s + (c + K_p)} \quad (2)$$

$$\frac{\Omega(s)}{T_d(s)} = \frac{-1}{(I + K_D)s + (c + K_p)} \quad (3)$$

From Routh-Hurwitz stability criteria, the system is stable when $(I+K_D)/(K_p+c)$ is positive.

		$I+K_D$
	α	$c+K_p$
1	$(I+K_D)/(c+K_p)$	0

Steady-state output responding to unit-step command

$$\omega_{ss} = \frac{K_p}{c + K_p} \quad (4)$$

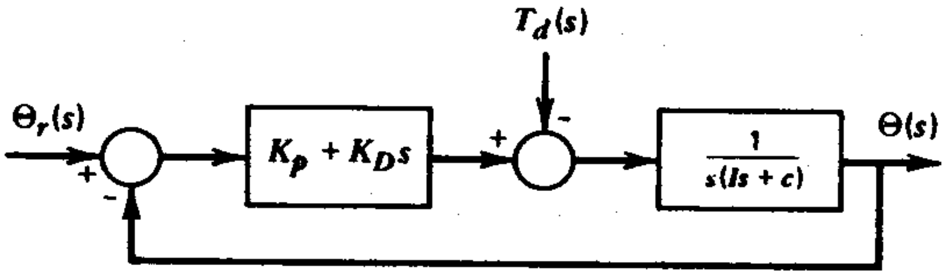
Steady-state error responding to unit-step disturbance

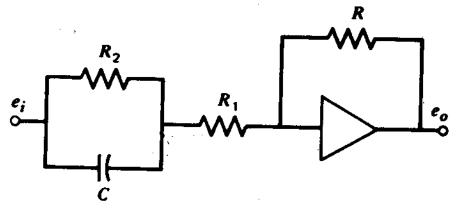
$$\omega_{ss} = -\frac{1}{c + K_p} \quad (5)$$

Time constant

$$\tau = \frac{I + K_D}{c + K_p} \qquad (6)$$

PD Control of a Second-Order System





$$e_o = -K_p\left(e_i + T_D\frac{de_i}{dt}\right) - \alpha T_D\frac{de_o}{dt}$$

$$K_p = \frac{R}{R_1 + R_2} \qquad T_D = R_2 C \qquad \alpha = \frac{R_1}{R_1 + R_2}$$

Figure 6.5-2 Simplified Form of Position Control System Using PD Control

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K_p + K_D s}{Is^2 + (c + K_D)s + K_p} \tag{6.5-5}$$

$$\frac{\Theta(s)}{T_d(s)} = \frac{-1}{Is^2 + (c + K_D)s + K_p} \tag{6.5-6}$$

- From Routh-Hurwitz stability criteria, the system is stable when $I/(c+K_D)$ and $(c+K_D)/K_p$ are positive.

		I	K_p
	α	$c+K_D$	0
1	$I/(c+K_D)$	K_p	0
2	$(c+K_D)/K_p$	0	0

- Steady-state output responding to unit-step command

$$\theta_{ss} = \frac{K_p}{K_p} = 1 \tag{6.5-7}$$

- Steady-state error responding to unit-step disturbance

$$\theta_{ss} = -\frac{1}{K_p} \tag{6.5-8}$$

$$Is^2 + (c+K_D)s + K_p = 0 \rightarrow s^2 + \frac{(c+K_D)}{I}s + \frac{K_p}{I} = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{6.5-9}$$

- Damping ratio

$$\zeta = \frac{c+K_D}{2\sqrt{IK_p}} \tag{6.5-10}$$

- Time constant

$$\tau = \frac{2I}{c+K_D} \tag{6.5-11}$$

Let's consider a general second-order system represented by its transfer function

$$G_p = \frac{1}{Is^2 + cs + k} \tag{1}$$

The closed loop transfer functions

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K_D s + K_p}{Is^2 + (c + K_D)s + (k + K_p)} \tag{2}$$

$$\frac{\Theta(s)}{T_d(s)} = \frac{-1}{Is^2 + (c + K_D)s + (k + K_p)} \tag{3}$$

From Routh-Hurwitz stability criteria, the system is stable when $I/(c+K_D)$ and $(c+K_D)/(k+K_p)$ are positive.

		I	$k+K_p$
	α	$c+K_D$	0
1	$I/(c+K_D)$	$k+K_p$	0
2	$(c+K_D)/(k+K_p)$	0	0

Steady-state output responding to unit-step command

$$\theta_{ss} = \frac{K_p}{k + K_p} \tag{4}$$

Steady-state error responding to unit-step disturbance

$$\theta_{ss} = -\frac{1}{k + K_p} \tag{5}$$

Time constant

$$\tau = \frac{2I}{c + K_D} \tag{6}$$

Damping ratio

$$\zeta = \frac{c + K_D}{2\sqrt{I(k + K_p)}} \tag{7}$$

6.6 PID Control

Let's consider a first-order system represented by its transfer function

$$G_p = \frac{1}{Is + c} \tag{1}$$

The closed loop transfer functions

$$\frac{\Omega(s)}{\Omega_r(s)} = \frac{K_D s^2 + K_p s + K_I}{(I + K_D)s^2 + (c + K_p)s + K_I} \tag{2}$$

$$\frac{\Omega(s)}{T_d(s)} = \frac{-s}{(I + K_D)s^2 + (c + K_p)s + K_I} \tag{3}$$

From Routh-Hurwitz stability criteria, the system is stable when $(I+K_D)/(c+K_p)$ and $(c+K_p)/(K_I)$ are positive.

		$I+K_D$	K_I
	α	$c+K_p$	0
1	$(I+K_D)/(c+K_p)$	K_I	0
2	$(c+K_p)/(K_I)$	0	0

Steady-state output responding to unit-step command

$$\omega_{ss} = 1 \tag{4}$$

Steady-state error responding to unit-step disturbance

$$\omega_{ss} = 0 \quad (5)$$

Time constant

$$\tau = \frac{2(I + K_D)}{c + K_p} \quad (6)$$

Damping ratio

$$\zeta = \frac{c + K_p}{2\sqrt{(I + K_D)K_I}} \quad (7)$$

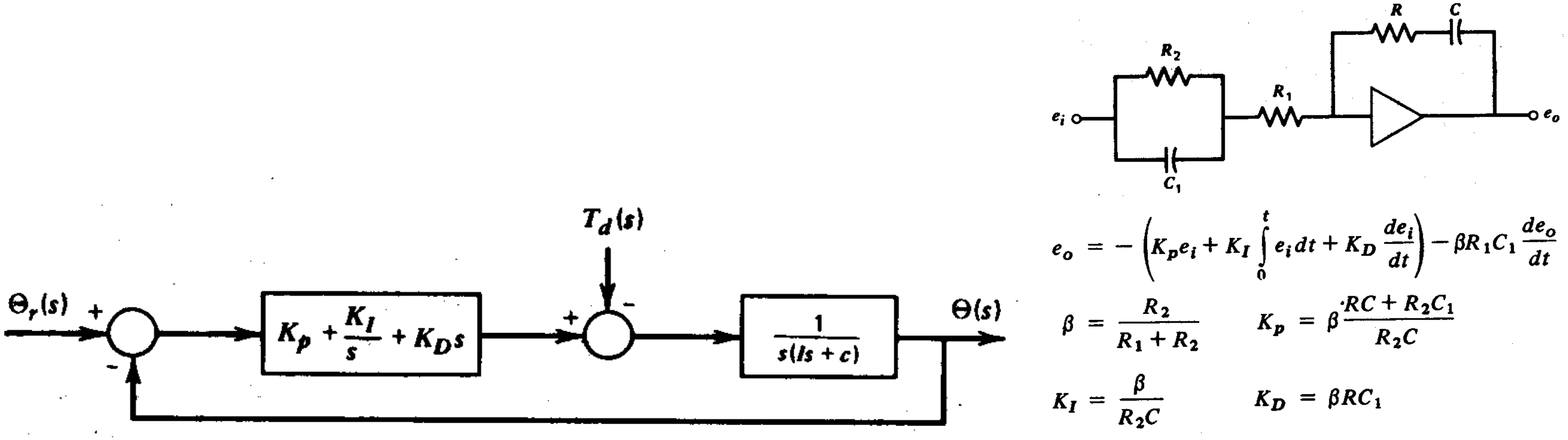


Figure 6.6-1 Simplified Form of Position Control System Using PID Control

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K_D s^2 + K_p s + K_I}{Is^3 + (c+K_D)s^2 + K_p s + K_I} \tag{6.6-1}$$

$$\frac{\Theta(s)}{T_d(s)} = \frac{-s}{Is^3 + (c+K_D)s^2 + K_p s + K_I} \tag{6.6-2}$$

- From Routh-Hurwitz stability criteria, the system stable is stable when $I/(c+K_D)$, $(c+K_D)^2/(K_p(c+K_D)-K_I I)$, and $(K_p(c+K_D)-K_I I)/((c+K_D)K_I)$ are positive.

		I	K_p
	α	$c+K_D$	K_I
1	$I/(c+K_D)$	$K_p-K_I I/(c+K_D)$	0
2	$(c+K_D)^2/(K_p(c+K_D)-K_I I)$	K_I	0
3	$(K_p(c+K_D)-K_I I)/((c+K_D)K_I)$	0	0

- Steady-state output responding to unit-step command

$$\theta_{ss} = 1 \tag{6.6-3}$$

- Steady-state error responding to unit-step disturbance

$$\theta_{ss} = 0 \tag{6.6-4}$$

Let's consider a general second-order system represented by its transfer function

$$G_p = \frac{1}{Is^2 + cs + k} \tag{1}$$

The closed loop transfer functions

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{K_D s^2 + K_p s + K_I}{Is^3 + (c + K_D)s^2 + (k + K_p)s + K_I} \tag{2}$$

$$\frac{\Theta(s)}{T_d(s)} = \frac{-s}{Is^3 + (c + K_D)s^2 + (k + K_p)s + K_I} \tag{3}$$

From Routh-Hurwitz stability criteria, the system stable is stable when $I/(c+K_D)$, $(c+K_D)^2/((k+K_p)(c+K_D)-K_I I)$, and $((k+K_p)(c+K_D)-K_I I)/((c+K_D)K_I)$ are positive.

		I	$k+K_p$
	α	$c+K_D$	K_I
1	$I/(c+K_D)$	$k+K_p-K_I I/(c+K_D)$	0
2	$(c+K_D)^2/((k+K_p)(c+K_D)-K_I I)$	K_I	0
3	$((k+K_p)(c+K_D)-K_I I)/((c+K_D)K_I)$	0	0

Steady-state output responding to unit-step command

$$\theta_{ss} = 1 \tag{4}$$

Steady-state error responding to unit-step disturbance $\theta_{ss} = 0$ (5)

7 Control System Design

Control Design Requirements

1. Equilibrium specifications
   (a) Stability
   (b) Steady-state error
2. Transient specifications
   (a) Speed of response
   (b) Form of response (degree of damping)
3. Sensitivity specifications
   (a) Sensitivity to parameter variations
   (b) Sensitivity to model inaccuracies
   (c) Noise rejection (bandwidth, etc.)
4. Nonlinear effects
   (a) Stability
   (b) Final control element capabilities

In addition to these performance stipulations, the usual engineering considerations of initial cost, weight, maintainability, and so forth must be taken into account.

Accurate plant model

- The range of adjustment of controller gains can usually be made small.
- This technique reduces the cost of the controller.

Non-accurate plant model

- A standard controller with several control modes and wide ranges of gains is used.
- This approach should be considered when the cost of developing an accurate plant model might exceed the cost of controller tuning in the field.

The Ziegler-Nichols Rules

- Ziegler and Nichols developed their rules to determine control gains for unknown plant model from experiments and by analyzing various industrial processes.
- The rules usually had a step response that was oscillatory but with enough damping so that the second overshoot was less than 25% of the first (peak) overshoot. This is the quarter-decay criterion.

- The first method, the ***process reaction method*** relies on the fact that many processes have an open-loop step response, process signature, characterized by two parameters, *R* and *L*.
- *R* is the slope of a line tangent to the steepest part of the response curve.
- *L* is the time at which this line interests the time axis.
- First- and second-order linear systems do not yield positive values for *L*, and so the method cannot be applied to such systems. However, third-order and higher linear systems with sufficient damping do yield such a response.

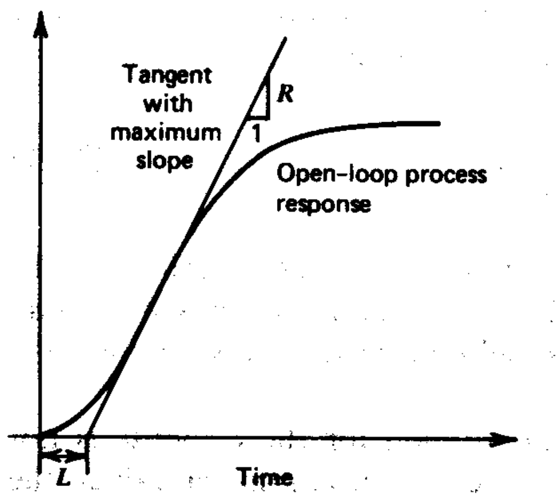


Figure 7-1 Process Signature for a Unit-Step Input. The parameters *R* and *L* are determined for use with the process-reaction method of Ziegler and Nichols.

Controller transfer function $G(s) = K_p\left(1 + \frac{1}{T_I s} + T_D s\right) = K_p + \frac{K_I}{s} + K_D s$

Control Mode	Process Reaction Method	Ultimate Cycle Method
P control	$K_p = 1/RL$	$K_p = 0.5\ K_{pu}$
PI control	$K_p = 0.9/RL$	$K_p = 0.45\ K_{pu}$
	$T_I = 3.3L$	$T_I = 0.83\ P_u$
PID control.	$K_p = 1.2/RL$	$K_p = 0.6\ K_{pu}$
	$T_I = 2L$	$T_I = 0.5\ P_u$
	$T_D = 0.5L$	$T_D = 0.125\ P_u$

Table 7-1 The Ziegler-Nichols Rules

- The ***ultimate-cycle method*** uses experiments with the controller in place. All control modes except proportional are turned off, and the process is started with the proportional gain K_p set at a low value. The gain is slowly increased until the process begins to exhibit sustained oscillations.
- Sustained oscillation occurs when a pair of roots is purely imaginary and the rest of the roots have negative real parts.
- Denote the period of this oscillation by P_u ($P_u = 2\pi/\omega_u$) and the corresponding ultimate gain by K_{pu}.
- Final tuning of the gains in the field is usually necessary.

8 Lead and Lag Compensation

8.1 Lead and Lag Compensator Circuits

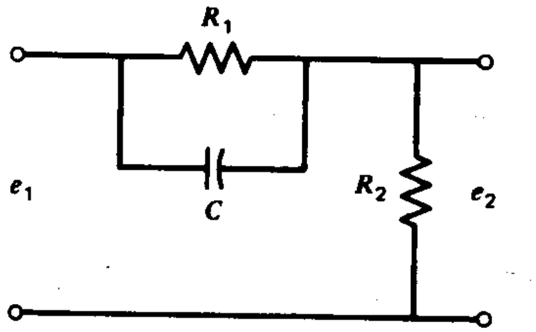


Figure 8.1-1 Passive-Lead Compensator

For lead compensator,

$$E_2(s) = I(s)R_2 = \frac{E_1(s)}{\dfrac{\dfrac{R_1}{sC}}{R_1 + \dfrac{1}{sC}} + R_2} R_2 \tag{8.1-1}$$

$$\frac{E_2(s)}{E_1(s)} = \frac{R_2 + R_1R_2Cs}{R_1 + R_2 + R_1R_2Cs} = \frac{1}{a}\frac{1+aTs}{1+Ts} = \frac{s+\dfrac{1}{aT}}{s+\dfrac{1}{T}} \tag{8.1-2}$$

where

$$a = \frac{R_1 + R_2}{R_2},\ a > 1 \text{ and } T = \frac{R_1R_2}{R_1 + R_2}C \tag{8.1-3}$$

Bode plots of the lead compensator,

$$\frac{s+\dfrac{1}{aT}}{s+\dfrac{1}{T}} = \frac{1}{a}\cdot\frac{\left(\dfrac{s}{1/(aT)}+1\right)}{\left(\dfrac{s}{1/T}+1\right)} \tag{8.1-4}$$

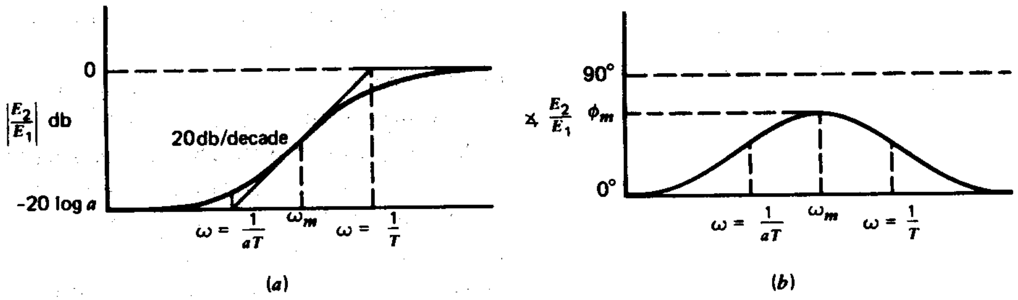


Figure 8.1-2 Bode Plots for the Lead Compensator

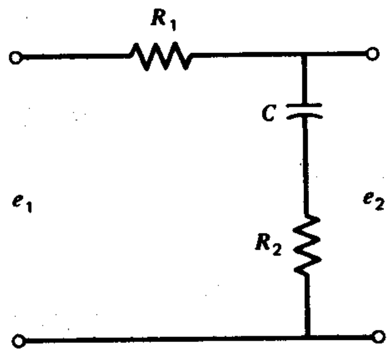


Figure 8.1-3 Passive-Lag Compensator

For lag compensator,

$$E_2(s) = I(s)\left(R_2 + \frac{1}{sC}\right) = \frac{E_1(s)}{R_1 + R_2 + \dfrac{1}{sC}}\left(R_2 + \frac{1}{sC}\right) \tag{8.1-5}$$

$$\frac{E_2(s)}{E_1(s)} = \frac{1 + R_2 Cs}{1 + (R_1 + R_2)Cs} = \frac{1 + aTs}{1 + Ts} = a\frac{s + \dfrac{1}{aT}}{s + \dfrac{1}{T}} \tag{8.1-6}$$

where

$$a = \frac{R_2}{R_1 + R_2},\ a < 1 \text{ and } T = C(R_1 + R_2) \tag{8.1-7}$$

Bode plots of the lag compensator,

$$a\frac{s+\frac{1}{aT}}{s+\frac{1}{T}}=\frac{\left(\frac{s}{1/(aT)}+1\right)}{\left(\frac{s}{1/T}+1\right)} \tag{8.1-8}$$

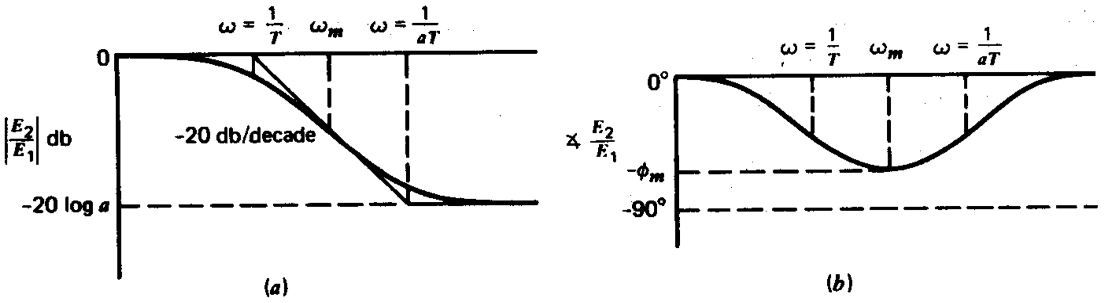
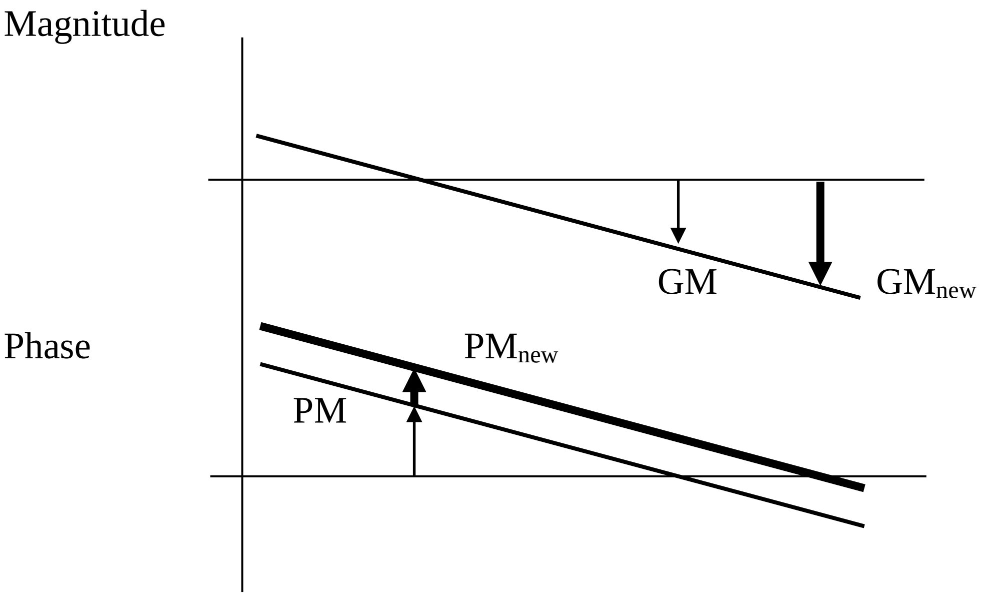


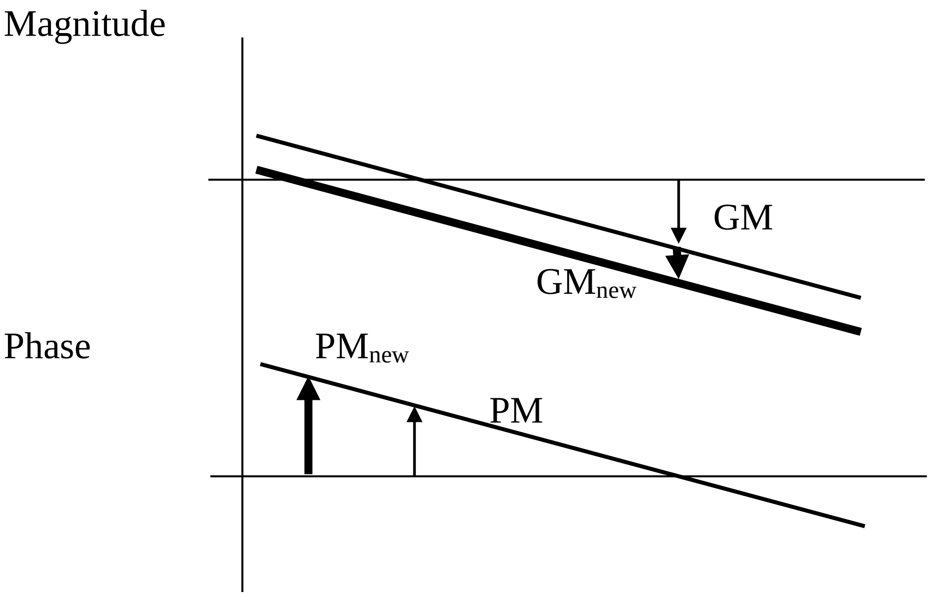
Figure 8.1-4 Bode Plots for the Lag Compensator

- When used in series with a proportional gain K_p, the lead compensator increases the gain margin. This allows the gain K_p/a to be made larger than is possible without the compensator. The result is a decrease in the closed-loop bandwidth and an increase in the speed of response.
- The lag compensator is used when the speed of response and damping of the closed-loop system are satisfactory, but the steady-state error is too large. The lag compensator allows the gain to be increased without substantially changing the resonance frequency ω_r and the resonance peak m_p of the closed-loop system.

Lead Compensator



Lag Compensator



8.2 Root Locus Design of Compensators

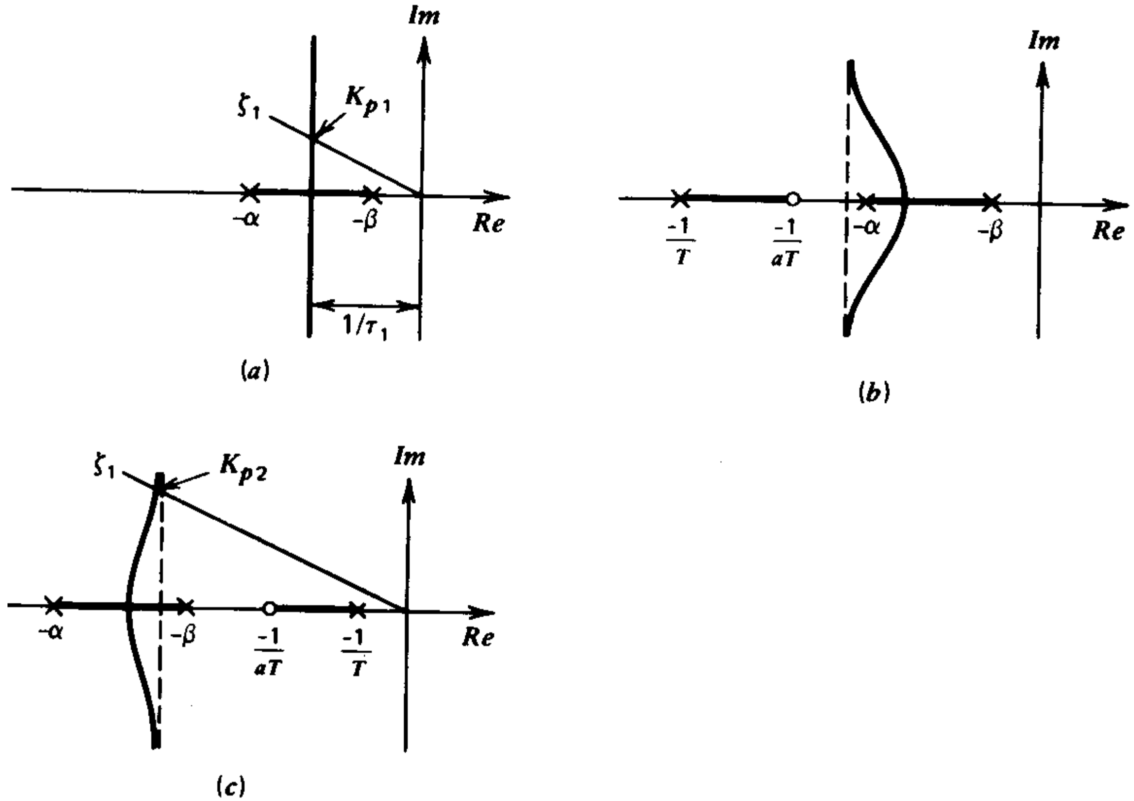


Figure 8.2-1 Effects of Series Lead and Lag Compensators (a) Uncompensated System's Root Locus (b) Root Locus with Lead Compensation (c) Root Locus with Lag Compensation

8.2.1 Lead Compensators

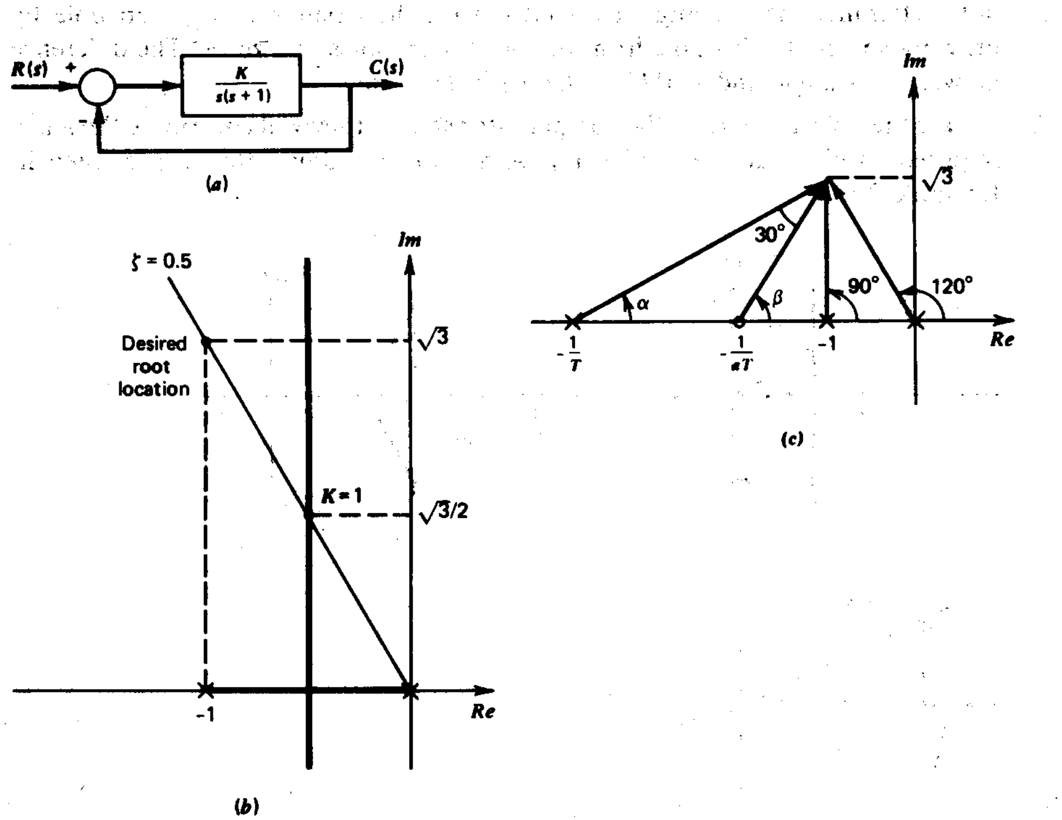
- With lead compensator, the pole and zero reshape the root locus so that a smaller dominant time constant can be obtained.
- Choosing the proportional gain high enough is required to place the roots close to asymptotes.
- Designing a lead compensator with the root locus is done as follows.

1. From the time domain specifications (time constant, damping ratio, etc.), the required locations of the dominant closed-loop poles are determined.
2. From the root locus plot of the uncompensated system, determine whether or not the desired closed-loop poles can be obtained by adjusting the open-loop gain. If not, determine the net angle associated with the desired closed-loop pole by drawing vectors to this pole from the open-loop poles and zeros. The difference between this angle and -180 ° is the angle deficiency.
3. Locate the pole and zero of the compensator so that they will contribute the angle required to eliminate the deficiency.
4. Compute the required value of the open-loop gain from the root locus plot.
5. Check the design to see if the specifications are met. If not, adjust the locations of the compensator's pole and zero.

Consider a system represented by

$$G(s) = \frac{1}{s(s+1)} \tag{1}$$

Block diagram and root locus of the controlled system are shown below.



Gain K = 1 gives damping ratio ζ = 0.5 and closed-loop poles at $s = -\frac{1}{2} \pm i\frac{1}{2}\sqrt{3}$, thus, time constant τ is 2 s. If transient specifications require ζ = 0.5 with the time constant of τ = 1. The closed-loop poles, thus, should be at $s = -1 \pm i\sqrt{3}$. Lead compensator is used. Phase diagram is considered as shown in diagram (c).

Characteristic equation of the compensated system is determined from return difference

$$1 + K\frac{(s + \frac{1}{Ta})}{(s + \frac{1}{T})} \cdot \frac{1}{s(s+1)} = 0 \qquad (2)$$

$$\frac{(s + \frac{1}{Ta})}{(s + \frac{1}{T})} \cdot \frac{1}{s(s+1)} = -\frac{1}{K} \qquad (3)$$

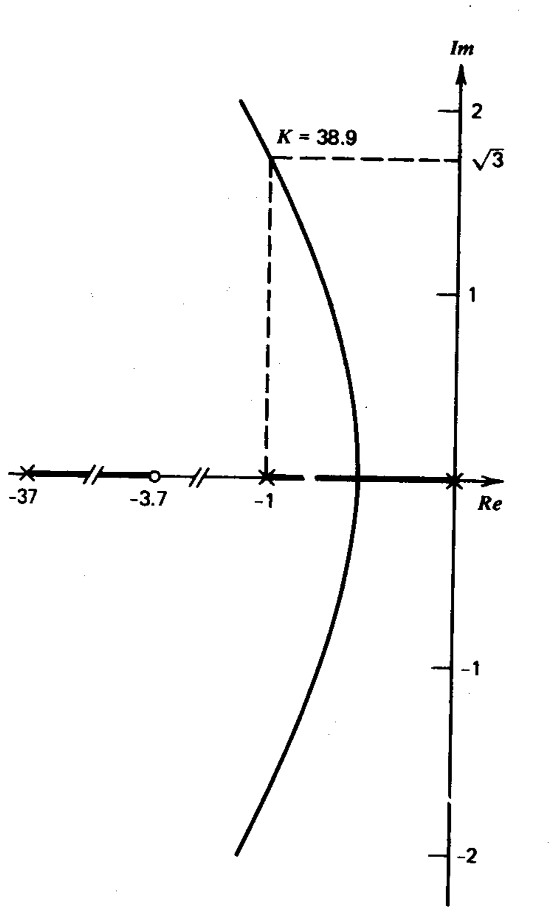
By applying phase equation

$$\angle(s + \frac{1}{Ta}) - \angle(s + \frac{1}{T}) - \angle(s) - \angle(s+1) = -180° \qquad (4)$$

$$\angle(s + \frac{1}{Ta}) - \angle(s + \frac{1}{T}) = (-180 + 120 + 90)° = 30° \qquad (5)$$

There are more unknown parameters than equations. If a is selected at 10, T = 0.027 is solved.

Root locus of the compensated system then becomes



The required gain, K, is 38.9.

8.2.2 Lag Compensators

- Suppose that the desired damping ratio ζ_1 and desired time constant τ_1 are obtainable with a proportional gain of K_{p1}, but the resulting steady-state error $\alpha\beta/(\alpha\beta + K_{p1})$ for a step input is too large. We need to increase the gain while preserving the desired damping ratio and time constant.
- With the lag compensator, the compensated system, roots with a damping ratio ζ_1, correspond to a high value of the proportional gain, K_{p2}.
- Thus, $K_{p2} > K_{p1}$, and the steady-state error will be reduced.

The effect of the lag compensator on the time constant can be seen as follows. The open-loop transfer function is

$$G(s)H(s) = \frac{aK_p\left(s + \frac{1}{aT}\right)}{(s+\alpha)(s+\beta)\left(s + \frac{1}{T}\right)} \tag{8.2.2-1}$$

If the value of T is chosen large enough, the pole at $s = -1/T$ is approximately canceled by the zero at $s = -1/aT$, and the open-loop transfer function is given approximately by

$$G(s)H(s) = \frac{aK_p}{(s+\alpha)(s+\beta)} \tag{8.2.2-2}$$

The compensation leaves the time constant relatively unchanged. Since $a < 1$, K_p can be selected as the larger value. The ratio of K_{p1} to K_{p2} is approximately given by the parameter a.

- Design of a lag compensator with the root locus as follows:

1. From the root locus of the uncompensated system, the gain K_{p1} is found that will place the roots at the locations required to give the desired relative stability and transient response.
2. Let K_{p2} denote the value of the gain required to achieve the desired steady-state performance. The parameter a is the ratio of these two gain values $a = K_{p1}/K_{p2} < 1$.
3. The value of T is then chosen large so that the compensator's pole and zero are close to the imaginary axis. This placement should be made so that the compensated locus is relatively unchanged in the vicinity of the desired closed-loop poles. This will be true if the angle contribution of the lag compensator is close to zero.
4. Locate the desired closed-loop poles on the compensated locus and set the open-loop gain so that the dominant roots are at this location (neglecting the existence of the compensator's pole and zero).
5. Check the design to see if the specifications are met. If not, adjust the locations of the compensator's pole and zero.

Consider again the system represented by

$$G(s) = \frac{1}{s(s+1)} \quad (1)$$

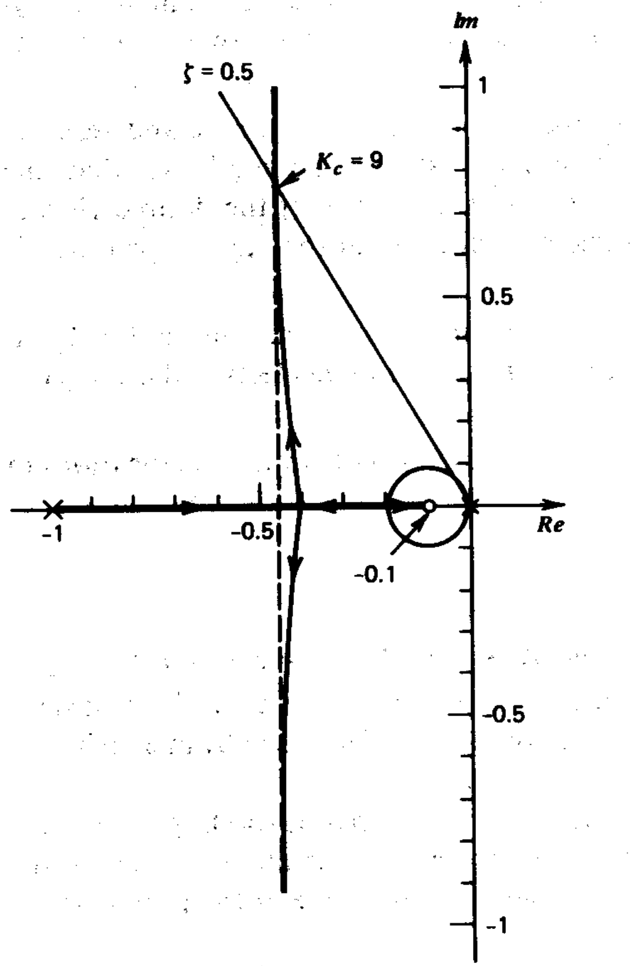
Assume the transient response at K = 1, τ =2 s and ζ = 0.5 and closed-loop poles at $s = -\frac{1}{2} \pm i\frac{1}{2}\sqrt{3}$, is acceptable. But steady-state error from command and/or disturbance is too high. (Steady-state error from unit-step command is 0 but steady-state error from unit-step disturbance is $-\frac{1}{K}$= -1.) If only about $\frac{1}{10}$ of the current steady-state error is required, thus the gain, K, should be about 10, likewise a should be about $\frac{1}{10}$.

$$\text{When } T \to \infty,\ s = 0, -\frac{1}{2} \pm i\frac{1}{2}\sqrt{3}, \text{ and } K = 10 \quad (2)$$

But this is impractical. There are more unknown parameters than equation, thus, a parameter must be selected. If poles at –0.01 and zero at –0.1 at K = 0 is selected.

$$T = 100,\ s = -0.111, -0.449 \pm i0.778, \text{ and } K = 9 \quad (3)$$

Root locus of the compensated system then become



8.3 Bode Design of Compensators

8.3.1 Lead Compensators

- For any particular value of the parameter *a*, the lead compensator can provide a maximum phase lead ϕ_m. This value, and the frequency ω_m at which it occurs, can be found as a function of *a* and *T* from the Bode plot.
- The frequency ω_m is the geometric mean of the two corner frequencies of the compensator. Thus,

$$\omega_m = \frac{1}{T\sqrt{a}} \tag{8.3.1-1}$$

By evaluating the phase angle of the compensator at this frequency,

$$\sin \phi_m = \frac{a-1}{a+1} \tag{8.3.1-2}$$

or

$$a = \frac{1+\sin \phi_m}{1-\sin \phi_m} \tag{8.3.1-3}$$

- System design in the frequency domain is usually done with phase and gain margin requirements.
- The purpose of the lead compensator is to use the maximum phase lead of the compensator to increase the phase of the open-loop system near the gain crossover frequency while not changing the gain curve near that frequency.

- Lead compensator design method:

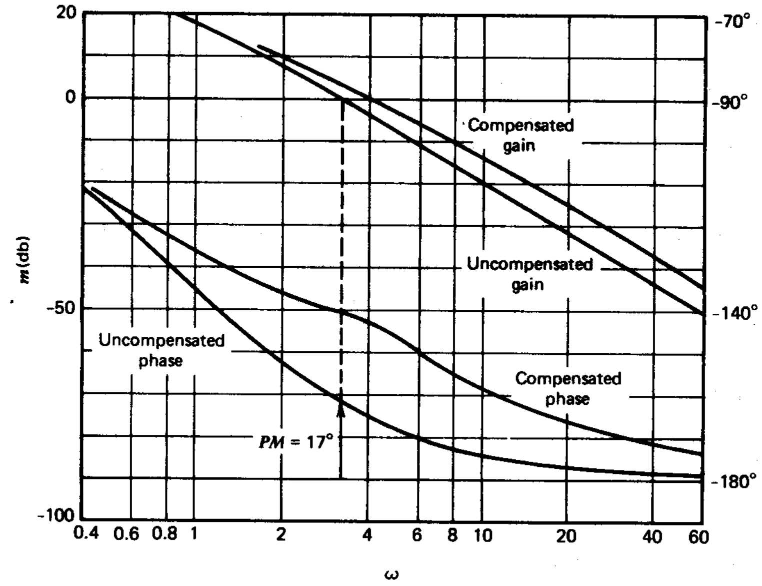
1. Set the gain K_p of the uncompensated system to meet the steady-state error requirement.
2. Determine the phase and gain margins of the uncompensated system from the Bode plot, and estimate the amount of phase lead ϕ required to achieve the margin specifications. The extra phase lead required can be used to estimate the value for ϕ_m to be provided by the compensator. Thus, a can be found from (8.3.1-3).
3. Choose T so that ω_m from (8.3.1-1) is located at the gain crossover frequency of the compensated system. One way of doing this is to find the frequency at which the gain of the uncompensated system equals $-20\log\sqrt{a}$. Choose this frequency to be the new gain crossover frequency. This frequency corresponds to the frequency ω_m at which ϕ_m occurs.
4. Construct the Bode plot of the compensated system to see if the specifications have been met. If not, the choice for ϕ_m needs to be evaluated and the process repeated. It is possible that a solution does not exist.

- The usual range for a is $1 < a < 20$.
- Additional phase lead can sometimes be obtained by cascading more than one lead compensator.

Consider again the system represented by

$$G(s) = \frac{1}{s(s+1)} \tag{1}$$

When K = 10 is selected, steady-state error meets the requirement but the transient performance is unsatisfactory. Assume gain margin of at least 6 db and a phase margin of at least 40° are required. Lead compensator will be designed.



From the Bode plot of the open-loop uncompensated system, the phase margin is 17° and the gain margin is infinite. The phase margin needs more ϕ_m = 40-17 = 23°. Thus

$$a = \frac{1+\phi_m}{1-\phi_m} = \frac{1+\sin 23°}{1-\sin 23°} = 2.283 \tag{2}$$

The new frequency where the phase margin happens is at the current magnitude of

$$-20\log\sqrt{a} = -3.585 \text{ db} \tag{3}$$

-3.585 db is at frequency, ω = 4 rad/s. Thus, *T* is determined from

$$T = \frac{1}{\omega_m\sqrt{a}} = \frac{1}{4\sqrt{2.283}} = 0.1655 \tag{4}$$

The compensator transfer function then should be

$$G_c(s) = \frac{s+\frac{1}{aT}}{s+\frac{1}{T}} = \frac{1}{a}\cdot\frac{aTs+1}{Ts+1} = \frac{1}{2.283}\cdot\frac{0.3778s+1}{0.1655s+1} \tag{5}$$

The open-loop transfer function of the compensated system is

$$G_c(s)G(s)H(s) = \frac{K(0.3778s+1)}{2.283s(s+1)(0.1655s+1)} \tag{6}$$

The original gain *K* = 10 no longer gives the required steady-state error because the compensator introduce the attenuation factor 1/2.283. The new gain must be *K* = 10(2.283) = 22.83 to achieve the required steady-state error. From the Bode plot the phase margin is 37° and the gain margin is still infinite. The trial and error can repeat in order to meet the requirement. ϕ_m should take into account this factor and can firstly be designed to 28°.

8.3.2 Lag Compensators

- Lag compensation uses the high-frequency attenuation of the network to keep the phase curve unchanged near the gain crossover frequency while this frequency is lowered.
- Lag compensator design procedure:

1. Set the open-loop gain K_p of the uncompensated system to meet the steady-state error requirements.
2. Construct the Bode plots for the uncompensated system, and determine the frequency at which the phase curve has the desired phase margin. Determine the number of decibels required at this frequency to lower the gain curve to 0 db. Let this amount be $m' > 0$ db and this frequency be ω'_g. Then a is found from

$$a = 10^{-m'/20} \tag{8.3.2-1}$$

3. The first two steps alter the phase curve. However, this curve will not be changed appreciably near ω'_g if T is chosen so that $\omega'_g >> 1/aT$. A good choice is to place the frequency $1/aT$ one decade below ω'_g. Any larger separation might result in a system with a slow response.
4. Construct the Bode plots of the compensated system to see if all the specifications are met. If not, choose another value for T and repeat the process.

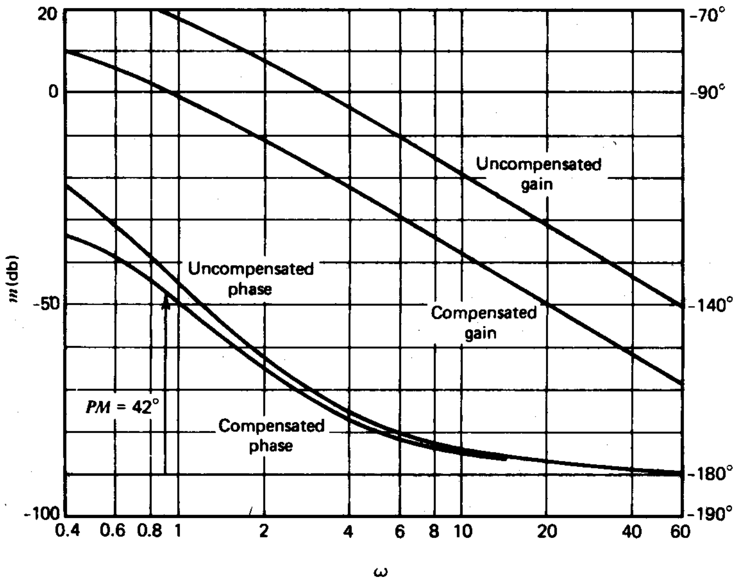
- A common range for a is $0.05 < a < 1$.
- The designer might add 5° to 10° to the specified phase margin before starting the design process.

Consider again the system represented by

$$G(s) = \frac{1}{s(s+1)} \tag{1}$$

When K = 1, the roots are $s = -0.5 \pm i0.866$. Assume this transient response is acceptable, but steady-state error requirement meets only when K = 10 which will alter the required transient response. Lag compensator is designed to improve the system to have the required steady-state error with gain margin of at least 6 db and a phase margin of at least 40°.

To achieve the required steady-state error K = 10 is required. Bode plot for the uncompensated open-loop system is shown.



The required phase margin after taking into account some factor is 40+5 = 45° This phase margin occurs at frequency ω = 1 rad/s. The gain at this frequency is 18 db, thus $m' = 18$ and

$$a = 10^{-m'/20} = 10^{-18/20} = 0.126 \tag{2}$$

T is selected to place ω = 1/(*aT*) one decade below ω = 1. Thus 1/(*aT*) = 0.1 and *T* = 79.4. The lag compensator results

$$G_c(s) = a\frac{s + \frac{1}{aT}}{s + \frac{1}{T}} = \frac{aTs + 1}{Ts + 1} = \frac{10s + 1}{79.4s + 1} \tag{3}$$

The open-loop transfer function of the compensated system is

$$G_c(s)G(s)H(s) = \frac{K(10s+1)}{s(s+1)(79.4s+1)} \tag{4}$$

With K = 10 the Bode plot is in the figure. The phase margin is 42° and the gain margin is infinite.

Summary

- Increasing phase curve will improve transient response.
- Increasing magnitude curve will improve steady-state response.

8.4 Lag-Lead Compensation

- The lead and lag compensators are complementary to each other in that one improves the transient performance while the other improves steady-state performance.

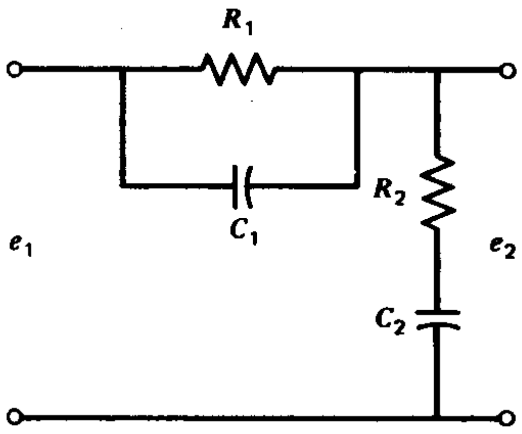


Figure 8.4-1 Passive Lag-Lead Compensator

For lead-lead compensator,

$$E_2(s) = I(s)\left(R_2 + \frac{1}{sC_2}\right) = \frac{E_1(s)}{\dfrac{\dfrac{R_1}{sC_1}}{R_1 + \dfrac{1}{sC_1}} + R_2 + \dfrac{1}{sC_2}}\left(R_2 + \frac{1}{sC_2}\right) \tag{8.4-1}$$

$$\frac{E_2(s)}{E_1(s)} = \frac{1 + (R_1C_1 + R_2C_2)s + R_1C_1R_2C_2s^2}{1 + (R_1C_1 + R_1C_2 + R_2C_2)s + R_1C_1R_2C_2s^2} = \frac{1 + aT_1s}{1 + T_1s}\frac{1 + bT_2s}{1 + T_2s} \tag{8.4-2}$$

where

$$aT_1 = R_1C_1,\ a > 1 \tag{8.4-3}$$

$$bT_2 = R_2C_2 \tag{8.4-4}$$

$$T_1 + T_2 = R_1C_1 + R_1C_2 + R_2C_2 \tag{8.4-5}$$

$$b = \frac{1}{a} \tag{8.4-6}$$

The Bode plots for this compensator for $T_2 > bT_2 > aT_1 > T_1$

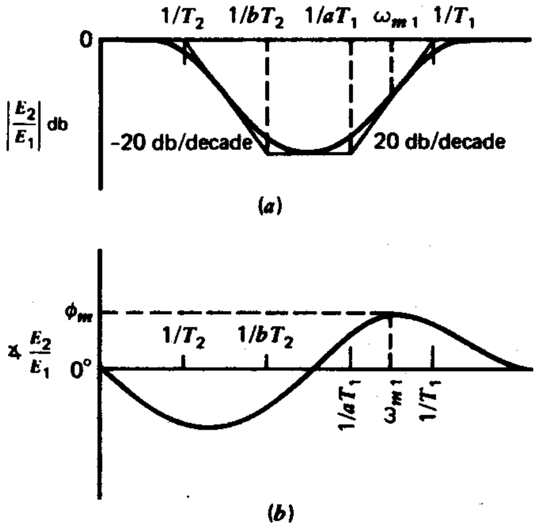


Figure 8.4-2 Bode Plots for the Lag-Lead Compensator

- The maximum phase shift ϕ_m occurs at $\omega_{m1} = 1/(T_1\sqrt{a})$.
- The attenuation at this frequency is equal in magnitude but opposite in sign to that of the lead compensator alone.
- The compensator affects the gain and phase in only the intermediate frequency range from $\omega = 1/T_2$ to $1/T_1$.

Let the compensator transfer function be written as

$$G_c(s) = G_1(s)G_2(s) \tag{8.4-7}$$

where $G_1(s)$ represents the lead compensation and $G_2(s)$ the lag compensation.

$$G_1(s) = \frac{1 + aT_1 s}{1 + T_1 s} \tag{8.4-8}$$

$$G_2(s) = \frac{1 + T_2 s / a}{1 + T_2 s} \tag{8.4-9}$$

The uncompensated open-loop transfer function is $G(s)H(s)$, so that of the compensated system is $G_c(s)G(s)H(s)$.

- The root locus approach to designing the lag-lead compensator is a combination of the approaches used for the lead and lag compensators.

1. Determine the desired locations of the dominant closed-loop poles from the transient performance specifications, and calculate the phase angle deficiency ϕ. This deficiency must be supplied by $G_1(s)$.
2. Use the steady-state error specifications to determine the required open-loop gain for $G_c(s)G(s)H(s)$.
3. Let s_d be the desired location of the dominant closed-loop poles. Assume the lag coefficient T_2 is large enough to achieve pole zero cancellation when s is near s_d; that is,

$$\left|G_2(s_d)\right| \cong 1 \tag{8.4-10}$$

With (8.4-10) satisfied, T_1 and a can be found from the magnitude criterion for the locus and the phase lead requirement from step 1; that is,

$$|G_1(s_d)||G(s_d)H(s_d)| = 1 \quad (8.4\text{-}11)$$

$$\angle G_1(s_d) = \phi \quad (8.4\text{-}12)$$

4. With the parameters a and T_1 now selected, choose T_2 so that condition (8.4-10) is satisfied. As before, check the design to see if the specifications are satisfied and the circuit elements are realizable.

- Designing a lag-lead compensator with the Bode plot follows the procedures for the lead and lag compensators discussed previously.
- Presumably the lag-lead is to be designed because of a deficiency in both the transient and steady-state performance of the uncompensated system.
- A designer who thinks that the transient response constitutes the most serious deficiency can choose to apply the procedures for the lead compensator first. When this part of the design is completed, the lag compensation can be designed.
- The opposite procedure could be used if the steady-state performance were worse than the transient.