Digital Control Theory

Advantage of using digital control over analog control

- Flexibility of the control algorithm
- Less expensive

Additional requirements

- Analog-to-digital (A/D) conversion
- Digital-to-analog (D/A) conversion

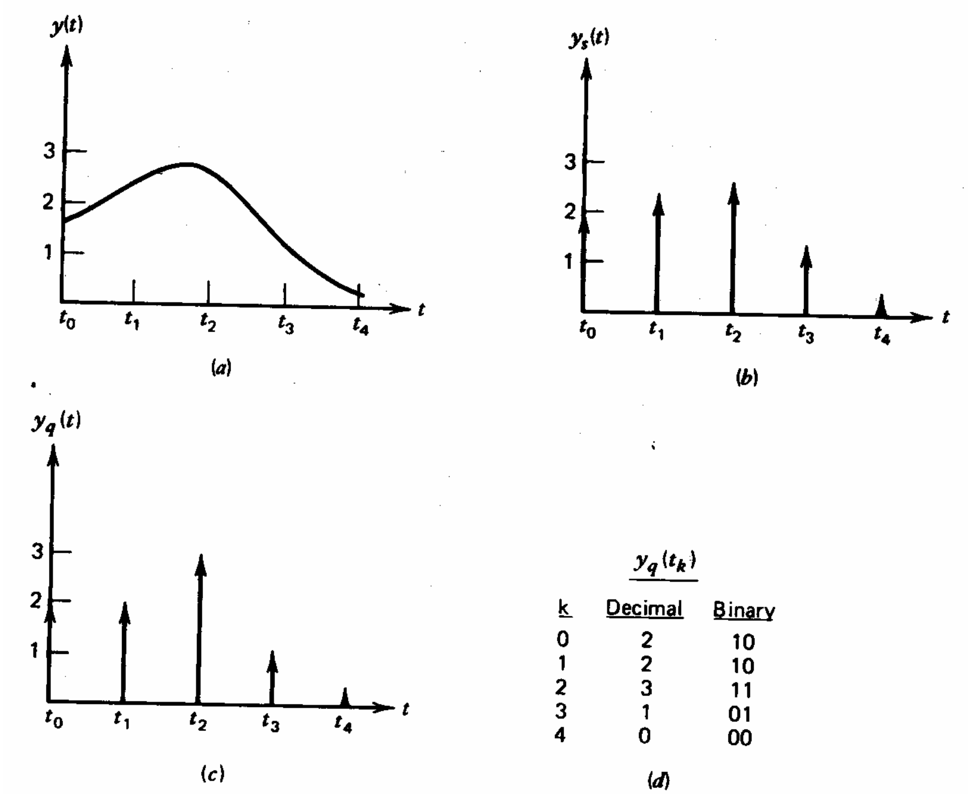
1 Discrete-Time Model

Analog-to-digital conversion: performed by A/D converter

Sampling: the process of measuring a continuous-time variable at distinct, separated instants of time.

Quantization: the process of rounding-off each pulse amplitude to one of a finite number of levels depending on the characteristics of the machine.

Coding: the process of converting the quantization level of each pulse into an equivalent binary number that the digital device can accept and store. The number of binary digits carried by the machine is its ***word length***, and this is obviously an important characteristic related to the device's **resolution**.



k	Decimal $y_q(t_k)$	Binary $y_q(t_k)$
0	2	10
1	2	10
2	3	11
3	1	01
4	0	00

Figure 1-1 Sampling, Quantization, and Coding of an Analog Signal. (a) Original Analog Signal as a Function of Time. (b) Sampled Signal. (c) Quantized Signal for a Hypothetical device with two bits. (d) Decimal-Coded and Binary-Coded Representations of the Quantized Signal.

1.1 Numerical Solution of Continuous-Time Models

There are several algorithms for obtaining difference equations from differential equations.

- Forward Euler method

$$\frac{d}{dt}y(t-\Delta t) \cong \frac{y(t)-y(t-\Delta t)}{\Delta t} \tag{1.1-1}$$

- Backward Euler method

$$\frac{d}{dt}y(t) \cong \frac{y(t)-y(t-\Delta t)}{\Delta t} \tag{1.1-2}$$

- Bilinear, Trapezoidal, or Tustin method

$$\frac{\frac{d}{dt}y(t)+\frac{d}{dt}y(t-\Delta t)}{2} \cong \frac{y(t)-y(t-\Delta t)}{\Delta t} \rightarrow s \cong \frac{2}{\Delta t}\cdot\frac{z-1}{z+1} \text{ and } z \cong \frac{1+s\Delta t/2}{1-s\Delta t/2} \tag{1.1-3}$$

Apply the forward Euler method applied to a first-order model

$$\frac{dy}{dt} = ry \text{ , } r = \text{constant} \tag{1.1-4}$$

$$\frac{dy}{dt} \cong \frac{y(t+\Delta t)-y(t)}{\Delta t} \tag{1.1-5}$$

$$y(t+\Delta t) = y(t) + ry(t)\Delta t \tag{1.1-6}$$

$$y(t_{k+1}) = (1+r\Delta t)y(t_k) \text{ or } y(k+1) = (1+r\Delta t)y(k) \tag{1.1-7}$$

where $y(k)$ represents $y(t)$ evaluated at $t = t_k$, $k = 0, 1, 2, \ldots$.

1.2 Free Response

Consider a linear first-order difference equation

$$y(k) = ay(k-1) \qquad k = 1, 2, 3, \ldots \tag{1.2-1}$$

$$y(1) = ay(0) \tag{1.2-2}$$

$$y(2) = ay(1) = a^2 y(0) \tag{1.2-3}$$

$$y(k) = a^k y(0) \qquad k = 0, 1, 2, 3, \ldots \tag{1.2-4}$$

The solution behavior of the difference equation depends on the value of a, it is summarized as followings for a positive $y(0)$.

1. $a > 1$. The solution's magnitude grows with time, and the solution keeps the sign of $y(0)$.
2. $a = 1$. The solution remains constant at $y(0)$.
3. $0 < a < 1$. The solution decays in magnitude and keeps the sign of $y(0)$.
4. $a = 0$. The solution jumps from $y(0)$ to zero at $k = 1$ and remains there.
5. $-1 < a < 0$. The magnitude decays, but the solution alternates sign at each time step (an oscillation with a period of two time units).
6. a = -1. The magnitude remains constant at $y(0)$, but the solution alternates sign at each step.
7. $a < -1$. The magnitude grows, and the sign of the solution alternates at each time step.

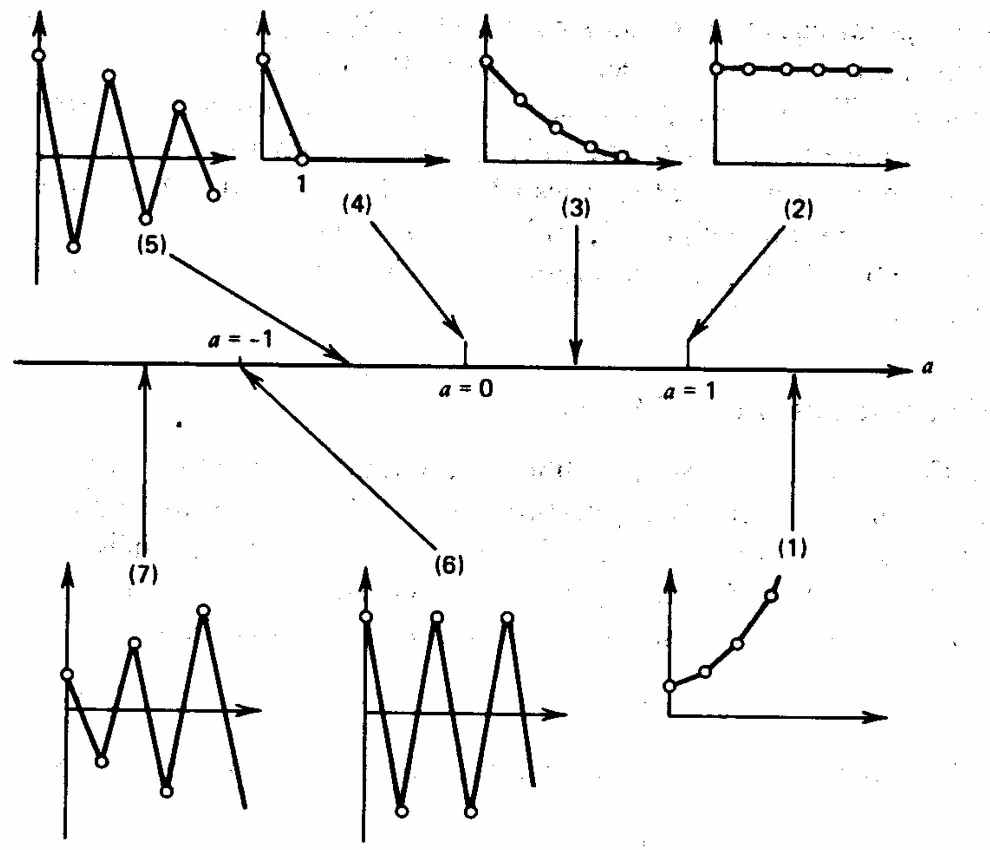


Figure 1.2-1 Free Response of the Linear First-Order Equation for Various Values of the Coefficient *a*.

1.3 Stability

- If $a \neq 1$, the only point equilibrium possible is $y_e = 0$.
- If $a = 1$, any value of y is an equilibrium.
- For an equilibrium at $y = 0$, the only cases in which the solution approaches and remains at equilibrium are the cases where $-1 < a < 1$.
- In other words, the model is stable if $|a| < 1$, unstable if $|a| > 1$, and neutrally stable if $|a| = 1$.

1.4 Relation to the Continuous-Time Model

Consider the first-order unforced continuous system,

$$\frac{dy}{dt} = ry \tag{1.4-1}$$

$$sY(s) - y(0) = rY(s) \tag{1.4-2}$$

$$(s-r)Y(s) = y(0) \tag{1.4-3}$$

$$Y(s) = \frac{1}{s-r}y(0) \tag{1.4-4}$$

$$y(t) = y(0)e^{rt} \tag{1.4-5}$$

If time t_k corresponds to the discrete time index k, then

$$y(t_k) = y(0)e^{rt_k} \tag{1.4-6}$$

The solution of discrete system,

$$y(t_k) = y(k) = a^k y(0) \tag{1.4-7}$$

$$e^{rt_k} = a^k \tag{1.4-8}$$

$$rt_k = \ln a^k = k\ln a \tag{1.4-9}$$

- A comparison between the two models can be made only if $a > 0$, since $\ln a$ is undefined for $a \leq 0$.
- Oscillatory behavior occurs in the discrete-time model only if $a < 0$.

Characteristic	Differential Equation $\dot{y} = ry$	Difference Equation $y(k) = ay(k-1)$
1. Solution	$y(t) = y(0)e^{rt}$	$y(k) = y(0)a^k$
2. Solution behavior	No oscillation	Oscillations of period two if $a < 0$
3. Stability		
Stable if	$r < 0$	$\|a\| < 1$
Neutrally stable if	$r = 0$	$\|a\| = 1$
Unstable if	$r > 0$	$\|a\| > 1$
4. Time constant (time to decay by 63%)	$t = \tau = -1/r,\ r < 0$	$k = -1/\ln \|a\|,\ \|a\| < 1$
5. Relation between the two models	(a) time t_k corresponds to time k (b) $rt_k = k \ln a$ if $a > 0$. (c) no relation if $a \leqslant 0$.	

Table 1.4-1 Comparison of Differential and Difference Equations

1.5 Sampling

- If the sampling frequency is not selected properly, the resulting sample sequence will not accurately represent the original signal.
- Sampling can be represented by the opening and closing of a switch.

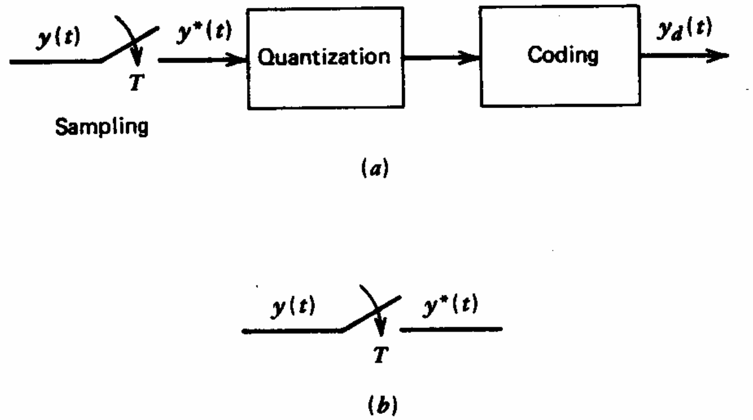


Figure 1.5-1 Block Diagram Representations of Analog-to-Digital Conversion. (a) Full Representation Showing the Sampling, Quanlization, and Coding Processes. (b) The Sampler Representation. This symbol is normally used instead of that shown in (a).

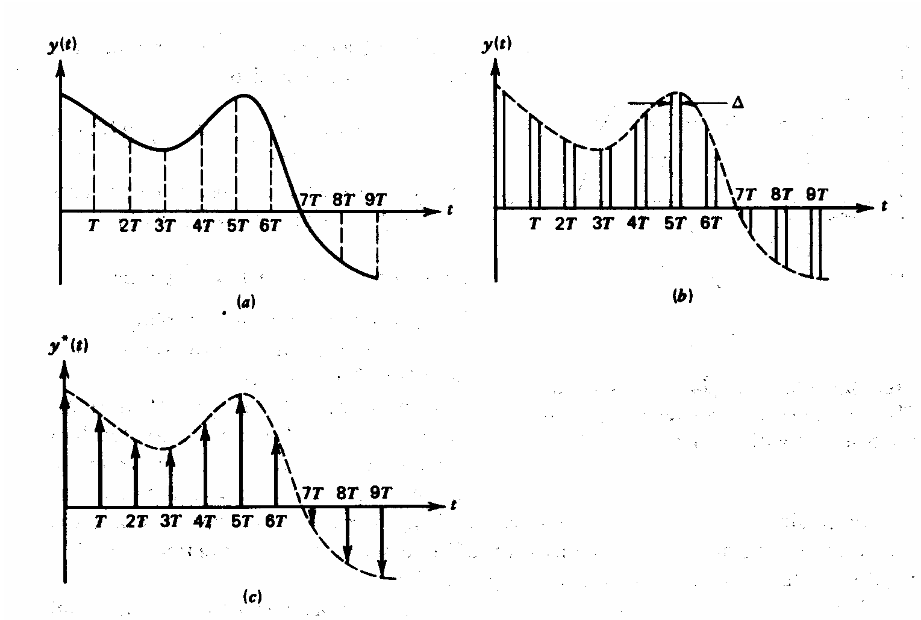


Figure 1.5-2 Sampling Process. (a) The Original Signal. (b) Illustration of the Sampling Duration Δ. (c) Impulse Representation of the Sampled Signal.

$$y^*(t) = y(0)\delta(t) + y(T)\delta(t-T) + y(2T)\delta(t-2T) + \cdots = \sum_{i=0}^{N} y(iT)\delta(t-iT) \quad (1.5\text{-}1)$$

$$y^*(kT) = \sum_{i=0}^{N-1} y(iT)\delta(kT - iT); \;\; y_k^* = \sum_{i=0}^{N-1} y_i \delta_{k-1}; \; y^*(k) = \sum_{i=0}^{N-1} y(i)\delta(k-i) \quad (1.5\text{-}2)$$

1.5.1 Aliasing

- Uniform sampling cannot distinguish between two sinusoidal signals when their circular frequencies have a sum or difference equal to $2\pi n/T$, where n is any positive integer, if the sampling rate is not high enough.
- The only effective frequency range for uniform sampling is $0 \le \omega \le \pi/T$.
- The frequency π/T (radians per unit time) is called the Nyquist frequency.
- This phenomenon is called ***aliasing***.

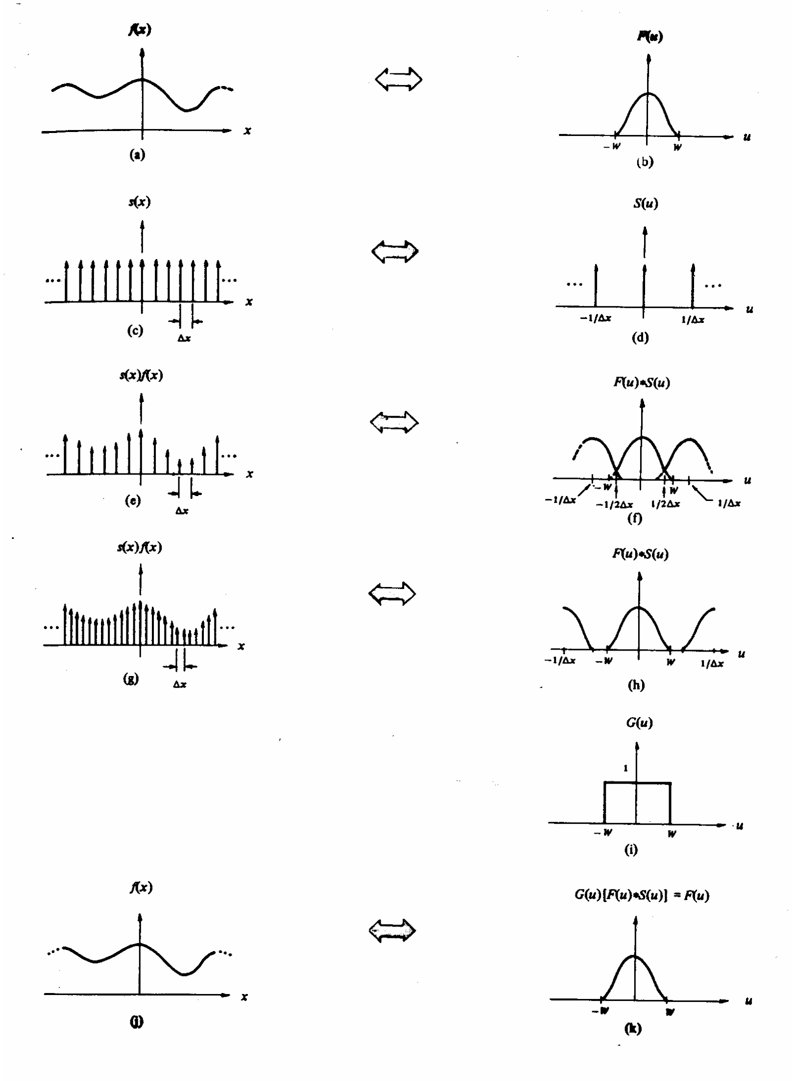
1.5.2 The Sampling Theorem

Sampling Theorem

- A continuous-time signal $y(t)$ can be reconstructed from its uniformly sampled values $y(kT)$ if the sampling period T satisfies

$$T \le \frac{\pi}{\omega_u} \quad (1.5.2\text{-}1)$$

  where ω_u is the highest frequency contained in the signal.
- Most physical signals have no finite upper frequency ω_u. Their spectra $|Y(\omega)|$ approach zero only as $\omega \to \infty$. In such cases, ω_u is estimated by finding the frequency range containing most of the signal's energy.


f(x)
F(u)
s(x)
S(u)
s(x)f(x)
F(u)*S(u)
G(u)
G(u)[F(u)*S(u)] = F(u)
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)

1.6 The Z Transform

$$y^*(t) = y(0)\delta(t) + y(T)\delta(t-T) + y(2T)\delta(t-2T) + \cdots \quad (1.6\text{-}1)$$

$$Y^*(s) = y(0) + y(T)e^{-Ts} + y(2T)e^{-2Ts} + \cdots \quad (1.6\text{-}2)$$

Define a new variable z as follows:

$$z = e^{Ts} \quad (1.6\text{-}3)$$

$$Y(z) = y(0) + y(T)\frac{1}{z} + y(2T)\frac{1}{z^2} + \cdots \quad (1.6\text{-}4)$$

- The transformed variable $Y(z)$ is the z transform of the function $y^*(t)$.
- $1/z$ is a delay operator represents a time delay T. $1/z^2$ represents a delay $2T$, and so forth.

Example To find the z transform of the sequence $y = \{1, a, a^2, a^3, \ldots\}$,

$$Y(z) = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots = \sum_{k=0}^{\infty} a^k z^{-k} \quad (1.6\text{-}5)$$

$$Y(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \quad (1.6\text{-}6)$$

Since $a + ab + ab^2 + \cdots = \frac{a}{1-b};\ b < 1$.

	$y(t)$, $t \geq 0$	$Y(s) = \int_0^\infty y(t)e^{-st}\,dt$	$Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k}$
1.	1	$\frac{1}{s}$	$\frac{z}{z-1}$
2.	t	$\frac{1}{s^2}$	$\frac{zT}{(z-1)^2}$
3.	$\frac{t^2}{2}$	$\frac{1}{s^3}$	$\frac{z(z+1)T^2}{2(z-1)^3}$
4.	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
5.	te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
6.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$
7.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$
8.	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$
9.	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2 - 2e^{-aT}\cos \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$

Table 1.6-1 Laplace and *Z* Transforms for Sampled Functions

	$y(k)$ for $k \geq 0$	$Y(z) = \sum_{k=0}^{\infty} y(k) z^{-k}$
1.	1	$\dfrac{z}{z-1}$
2.	a^k	$\dfrac{z}{z-a}$
3.	k	$\dfrac{z}{(z-1)^2}$
4.	k^2	$\dfrac{z(z+1)}{(z-1)^3}$
5.	ka^k	$\dfrac{az}{(z-a)^2}$
6.	$a^k \sin \omega k$	$\dfrac{az \sin \omega}{z^2 - 2az \cos \omega + a^2}$
7.	$a^k \cos \omega k$	$\dfrac{z(z - a \cos \omega)}{z^2 - 2az \cos \omega + a^2}$

Table 1.6-2 *Z* Transform Pairs

Property	$y(t)$ or $y(k)$	$\mathscr{Z}[y(t)]$ or $\mathscr{Z}[y(k)]$
1. Linearity	$ay(t)+bx(t)$	$aY(z)+bX(z)$
2. Right-shifting	$y(t-mT)$ $y(k-m)$	$z^{-m}Y(z)$
3. Left-shifting	(a) $y(t+T)$ $y(k+1)$	$zY(z)-zy(0)$
	(b) $y(t+2T)$ $y(k+2)$	$z^2Y(z)-z^2y(0)-zy(T)$ $z^2Y(z)-z^2y(0)-zy(1)$
	(c) $y(t+mT)$	$z^mY(z)-\sum_{i=0}^{m-1} y(iT)z^{m-i}$
	$y(k+m)$	$z^mY(z)-\sum_{i=0}^{m-1} y(i)z^{m-i}$
4. Differentiation	$ty(t)$	$-Tz\frac{d}{dz}[Y(z)]$
	$ky(k)$	$-z\frac{d}{dz}[Y(z)]$
5. Convolution	$\sum_{i=0}^{k} x(iT)y(kT-iT)$	$X(z)Y(z)$
6. Summation	$\sum_{i=0}^{k} y(iT)$	$\frac{z}{z-1}Y(z)$
7. Multiplication by an exponential	(a) $e^{-at}y(t)$	$Y(ze^{aT})$
	(b) $a^k y(k)$	$Y\left(\frac{z}{a}\right)$
8. Initial value theorem	$y(0)=\lim_{\lvert z\rvert\to\infty} Y(z)$	
9. Final value theorem	$y(\infty)=\lim_{z\to 1}\frac{(z-1)}{z}Y(z)$ if $\frac{(z-1)}{z}Y(z)$ is analytic for $\lvert z\rvert \geqslant 1$	

Table 1.6-3 Properties of the Z Transform

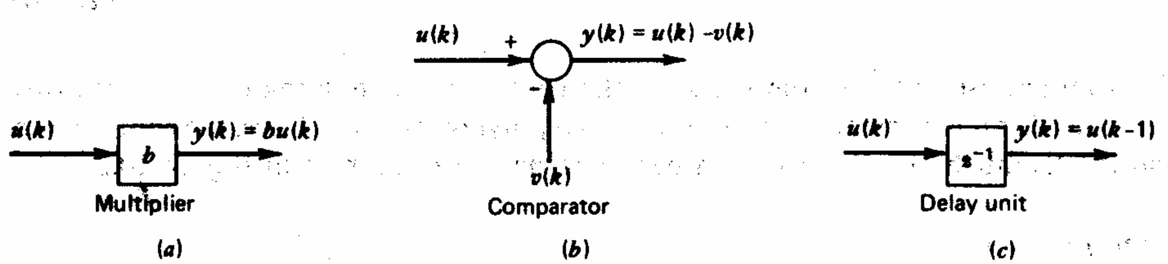
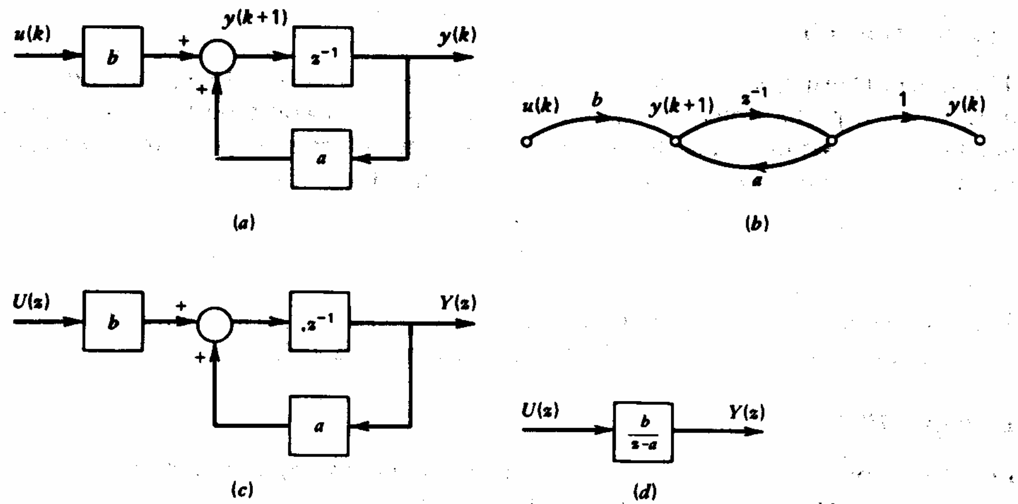


Figure 1.6-1 Three Basic Elements of Block Diagrams (a) Multiplier (b) Comparator (c) Delay Unit



$$y(k+1) = bu(k) + ay(k)$$
$$Yz = bU + aY$$
$$Y(z-a) = bU$$
$$\frac{Y}{U} = \frac{b}{z-a}$$

Figure 1.6-2 Representation of a First-Order System (a) Block Diagram in the Time Domain (b) Signal Flow Graph (c) Block Diagram in Terms of *Z*-Transform (d) Reduced Block Diagram Showing the System Transfer Function

1.7 Sampled-Data Systems

- If a digital device is used to measure or control the mechanical element, the resulting system is a discrete-continuous hybrid, ***sampled-data***, system.

1.7.1 Zero-Order Hold Circuit

Digital-to-analog converter performs two functions:

- Generation of the output pulses from the digital representation produced by the machine
- Conversion of the pulses to analog form

A zero-order hold circuit converts an impulse sequence into a continuous signal by holding the value of the impulse until the next pulse, duration *T*, the sampling period.

$$G(s) = \frac{1}{s}(1 - e^{-Ts}) \tag{1.7.1-1}$$

$$Y(s) = G(s)U^*(s) \tag{1.7.1-2}$$

- Each impulse at the input to the zero-order hold is converted into a rectangular pulse of width *T* and a height $u(kT)$ equal to the sample value at that time.
- If these pulses are applied to an element whose time constants are large compared to *T*, the pulses can be considered to be impulses with a strength $Tu(kT)$.

- The term zero order refers to the zero-order polynomial used to extrapolate between the sampling times.
- A first-order hold uses a first-order polynomial (a straight line with nonzero slope) for extrapolation.

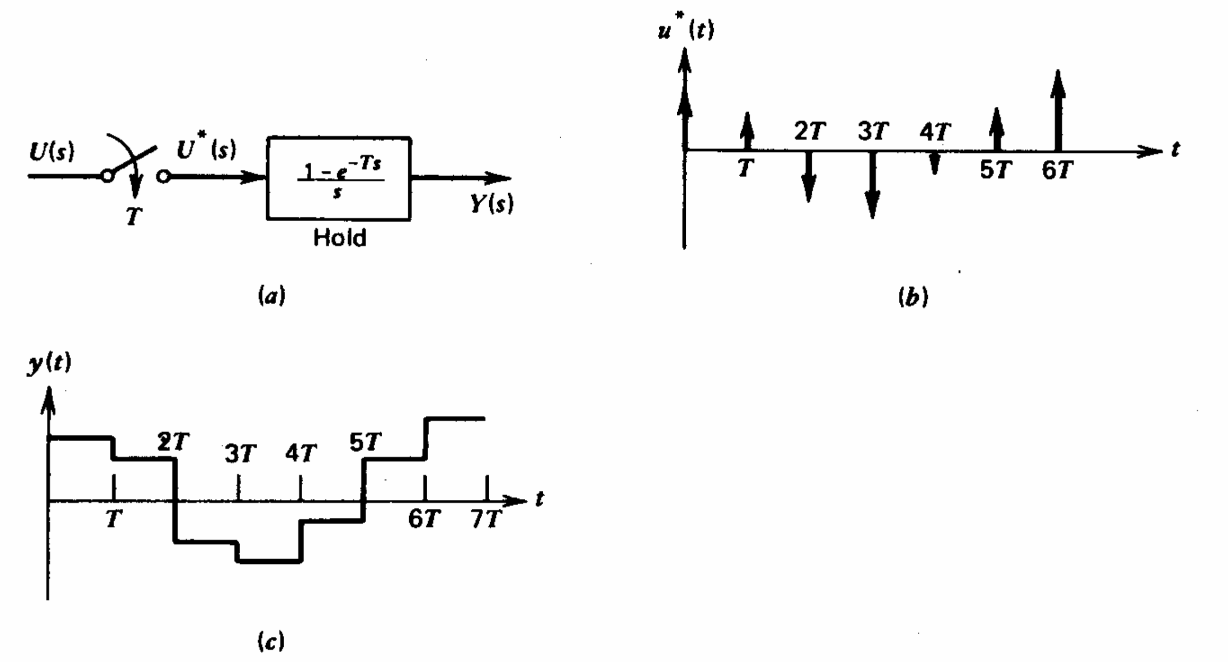


Figure 1.7.1-1 Zero-Order Hold (a) Block Diagram Representation of the Hold with a Sampler (b) Sampled Input Sequence (c) Resulting Analog Output from the Hold for the Input Sequence Shown in (b)

1.7.2 Pulse Transfer Functions of Analog Elements

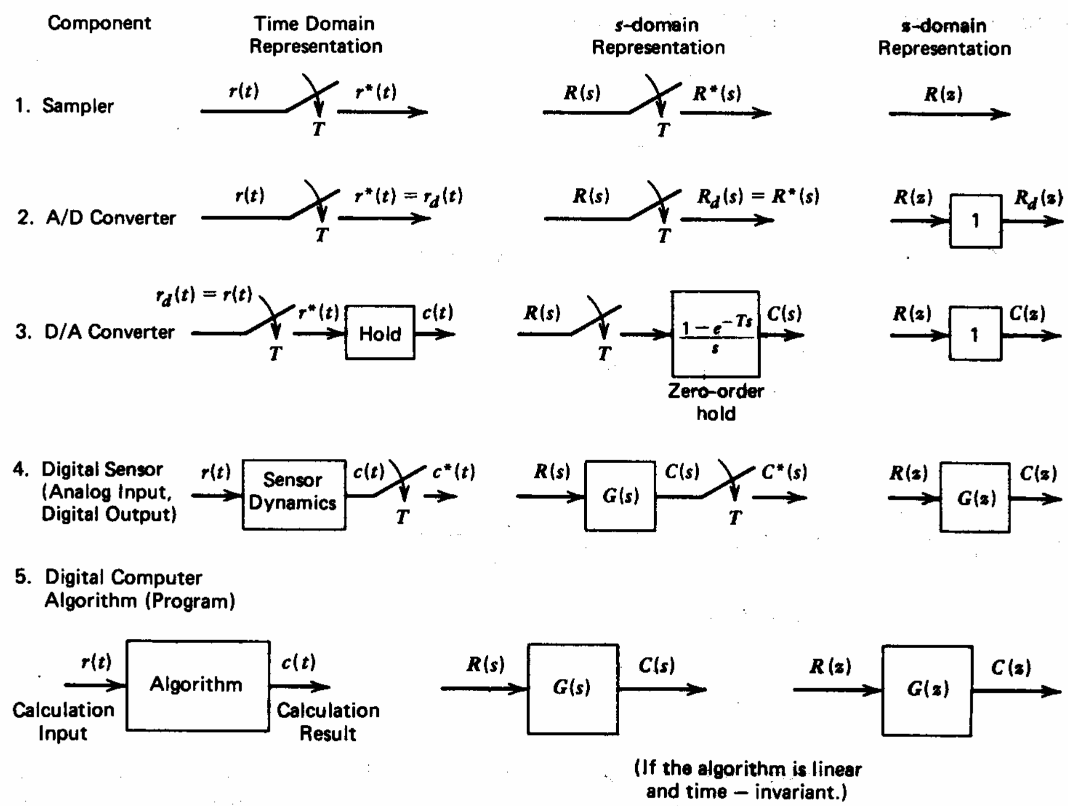


Figure 1.7.2-1 Representations of Common Digital Elements.

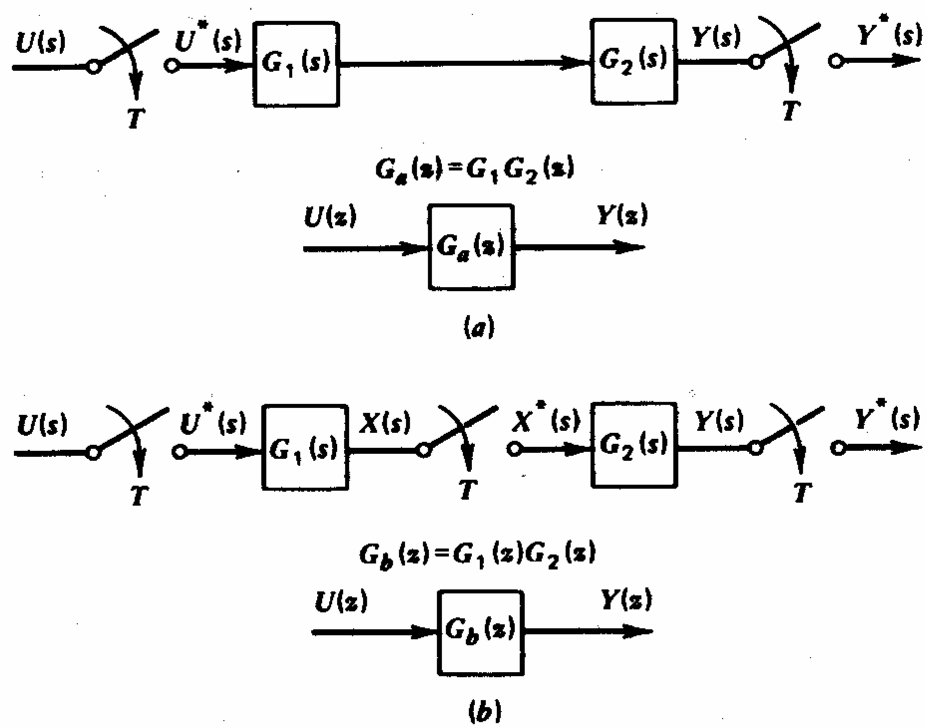
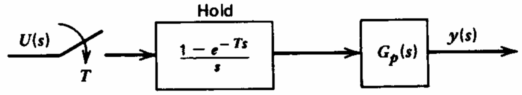


Figure 1.7.2-2 Transfer Functions of Cascaded Elements (a) Analog Elements Not Separated by a Sampler (b) Analog Elements Separated by a Sampler

s **domain**

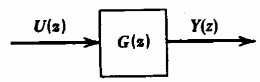


$$\frac{Y(s)}{U(s)} = G(s) = \frac{1-e^{-Ts}}{s} G_p(s) = (1-e^{-Ts})G_2(s)$$

where

$$G_2(s) = \frac{G_p(s)}{s}$$

z **domain**



$$\frac{Y(z)}{U(z)} = G(z) = \frac{z-1}{z} G_2(z)$$

Table 1.7.2-1 Transfer Function and the Zero-Order Hold

If the hold element is cascaded with an analog element with no sampler in between,

$$G(s) = G_1(s)G_2(s) \tag{1.7.2-1}$$

where $G_1(s) = 1 - e^{-Ts}$ and $G_2(s)$ is the remainder of $G(s)$.

$$G(s) = G_2(s) - e^{-Ts}G_2(s) \tag{1.7.2-2}$$

$$G(z) = G_2(z) - z^{-1}G_2(z) = (1-z^{-1})G_2(z) = \frac{z-1}{z}G_2(z) \tag{1.7.2-3}$$

1.8 Stability Tests

- The transformation that maps the inside of the unit circle in the *z* plane onto the entire left half of the *s* plane,

$$z = \frac{s+1}{s-1} \tag{1.8-1}$$

- Substitute *z* from (1.8-1) into the characteristic equation in terms of *z*. This gives a polynomial in *s* to which the Routh-Hurwitz criterion can be applied.

Characteristic Equation	Stability Requirements	
1. $F(z) = b_1 z + b_0 = 0$	$b_1 > \lvert b_0 \rvert$	**(1.1)**
$b_1 > 0$		
2. $F(z) = b_2 z^2 + b_1 z + b_0 = 0$	$b_2 + b_1 + b_0 > 0$	**(2.1)**
$b_2 > 0$	$b_2 - b_1 + b_0 > 0$	**(2.2)**
	$b_2 > \lvert b_0 \rvert$	**(2.3)**
3. $F(z) = b_3 z^3 + b_2 z^2 + b_1 z + b_0 = 0$	$b_3 + b_2 + b_1 + b_0 > 0$	**(3.1)**
$b_3 > 0$	$b_3 - b_2 + b_1 - b_0 > 0$	**(3.2)**
	$b_3 > \lvert b_0 \rvert$	**(3.3)**
	$\lvert b_0^2 - b_3^2 \rvert > \lvert b_0 b_2 - b_1 b_3 \rvert$	**(3.4)**

Table 1.8-1 Jury's Stability Results for First-, Second-, and Third-Order Systems

1.9 Transient Performance Specifications

$$z = e^{sT} \tag{1.9-1}$$

Consider a stable root pairs $s = -a \pm ib$.

$$z = e^{-aT}e^{\pm ibT} = e^{-aT}(\cos bT \pm i \sin bT) \tag{1.9-2}$$

For a fixed damping ratio $\zeta = \cos\beta$,

$$s = -a + ib = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2} \tag{1.9-3}$$

$$z = e^{-\zeta\omega_n T}e^{i\omega_n T\sqrt{1-\zeta^2}} \tag{1.9-4}$$

$$z = e^{sT} = re^{i\theta}, r > 0 \tag{1.9-5}$$

when $r = e^{-\zeta\omega_n T}, (\ln r = -\zeta\omega_n T)$, and $\theta = \omega_n T\sqrt{1-\zeta^2}$.

$$\frac{\ln r}{\theta} = -\frac{\zeta}{\sqrt{1-\zeta^2}} \tag{1.9-6}$$

$$\zeta = -\frac{\ln r}{\sqrt{\theta^2 + \ln^2 r}} \tag{1.9-7}$$

$$\omega_n = -\frac{\ln r}{\zeta T} = \frac{1}{T}\sqrt{\theta^2 + \ln^2 r} \tag{1.9-8}$$

$$\tau = \frac{1}{\zeta\omega_n} = -\frac{T}{\ln r} \tag{1.9-9}$$

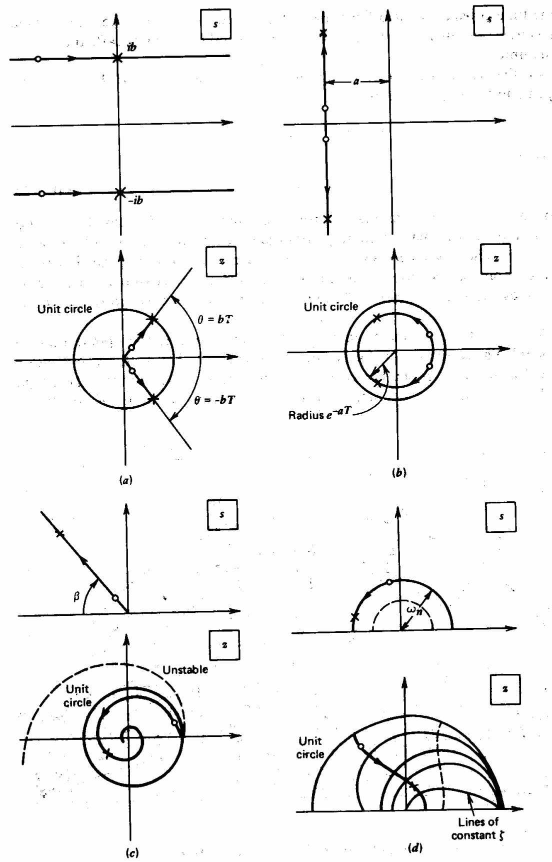


Figure 1.9-1 Equivalent Root Paths in *s* Plane and *z* Plane (a) Roots with the Same Oscillation Frequency *b*, (b) Roots with the Same Time Constant $\tau = 1/a$, (c) Roots with the Same Damping Ratio ζ, (d) Roots with the Same Natural Frequency ω_n

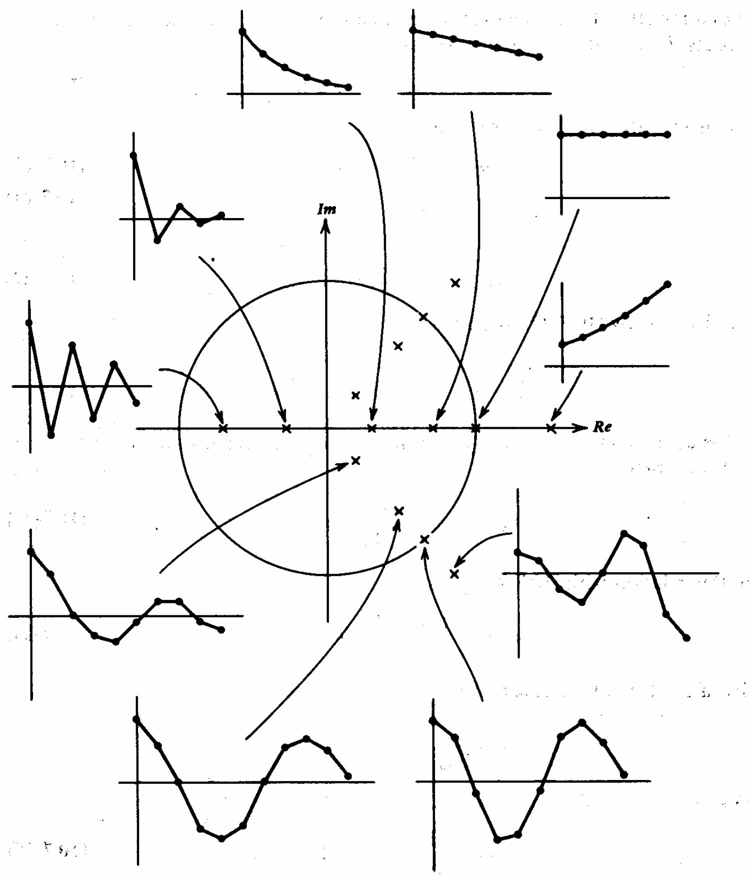


Figure 1.9-2 Free Response as a Function of Root Location in the *z* Plane

2 Digital Control

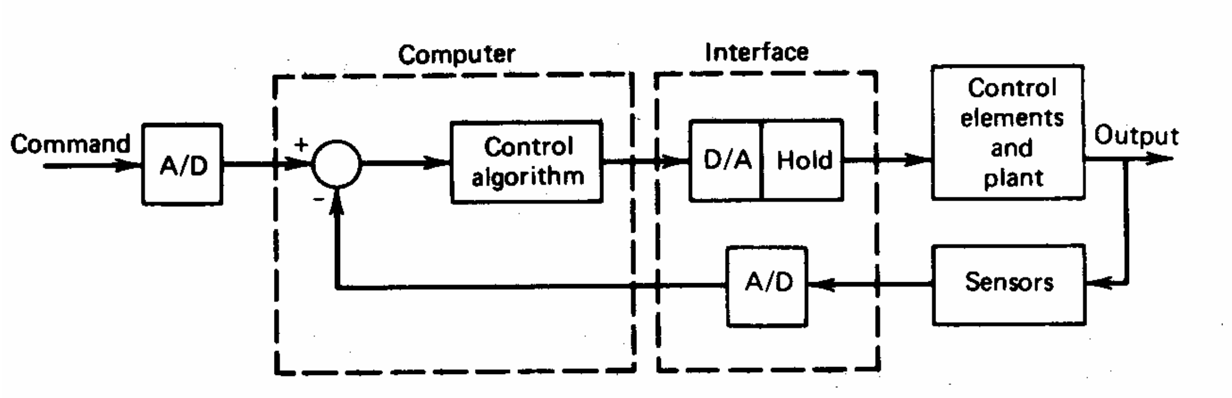


Figure 2-1 Structure of Computer Control System

2.1 Digital Control of a Motor

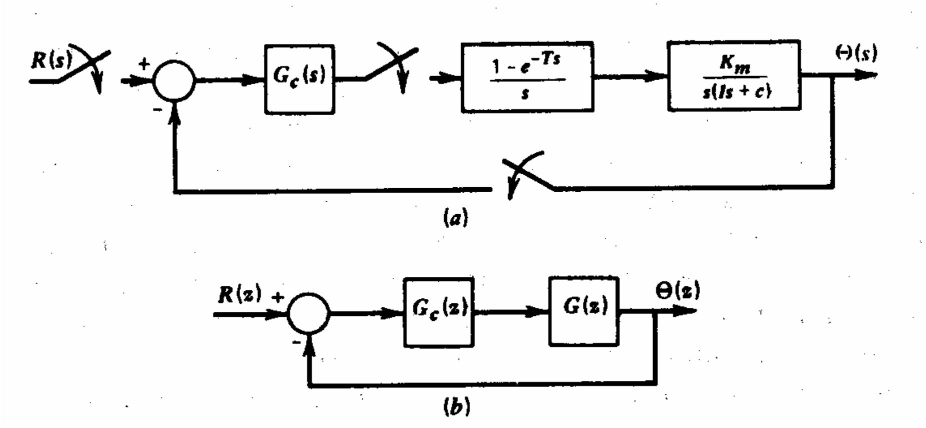


Figure 2.1-1 Digital Control of a DC Motor

(a) Sampled Data Diagram in the *s* Domain (b) Diagram in the *z* Domain

$$G(z) = \frac{z-1}{z} G_2(z) = \frac{\Theta(z)}{E(z)} \quad (2.1\text{-}1)$$

$$G_2(s) = \frac{K_m}{s^2(Is+c)} = \frac{K}{s^2(s+b)} \quad (2.1\text{-}2)$$

where $K = K_m / I, b = c / I$.

$$G_2(s) = \frac{K}{b}\left[\frac{1}{s^2} - \frac{1}{b}\left(\frac{1}{s} - \frac{1}{s+b}\right)\right] \tag{2.1-3}$$

$$G_2(z) = \frac{K}{b}\left[\frac{zT}{(z-1)^2} - \frac{1}{b}\left(\frac{z}{z-1} - \frac{z}{z-a}\right)\right] \tag{2.1-4}$$

where $a = e^{-bT}$ and T is the sampling period.

$$G(z) = \frac{K}{b^2}\frac{(bT-1+a)z+1-a-bTa}{(z-1)(z-a)} \tag{2.1-5}$$

$$G(z) = K\frac{b_1 z + b_2}{(z-1)(z-a)} \tag{2.1-6}$$

$$T(z) = \frac{\Theta(z)}{R(z)} = \frac{G_c(z)G(z)}{1+G_c(z)G(z)} = \frac{KG_c(z)(b_1 z + b_2)}{(z-1)(z-a)+KG_c(z)(b_1 z + b_2)} \tag{2.1-7}$$

2.2 Position and Velocity Algorithm

The position versions of proportional-plus-sum and proportional-plus-difference algorithms of the digital control law,

$$f(k) = K_p e(k) + K_I T \sum_{i=0}^{k} e(i) \tag{2.2-1}$$

$$f(k) = K_p e(k) + \frac{K_D}{T}[e(k) - e(k-1)] \tag{2.2-2}$$

where $f(k)$ and $e(k)$ are the control and error signals and T is the sampling period.

$$F(z) = \left(K_p + K_I T \frac{z}{z-1} \right) E(z) \tag{2.2-3}$$

$$F(z) = \left(K_p + \frac{K_D}{T}(1 - z^{-1}) \right) E(z) \tag{2.2-4}$$

The incremental or velocity versions of the algorithms determine the change in the control signal $f(k)$ - $f(k$-1).

$$f(k) = f(k-1) + K_p[e(k) - e(k-1)] + K_I T e(k) \tag{2.2-5}$$

$$f(k) = f(k-1) + K_p[e(k) - e(k-1)] + \frac{K_D}{T}[e(k) - 2e(k-1) + e(k-2)] \tag{2.2-6}$$

$$F(z) = \frac{(K_p + K_I T)z - K_p}{z-1} E(z) \tag{2.2-7}$$

$$F(z) = \frac{(K_p T + K_D)z^2 - (K_p T + 2K_D)z + K_D}{Tz(z-1)} E(z) \tag{2.2-8}$$

Advantages of the increment version over position version

- Maintaining of system last position in the event of failure or shutdown of the control computer
- No saturation at start-up if the controller is not matched to the current position
- Well suit with incremental output devices, such as stepper motors

Problems of the conventional digital controller

- Consider the integral term of velocity version, $K_I T e(k)$. If T and $e(k)$ are small, the finite word length of the machine can result in a zero change in the integral output. Nonzero error causes no change in the control signal results in a steady-state offset error, which never occurs with I action in analog systems.
- Solutions
  - Improve the resolution by increasing the word length of the computer.
  - Before the integral is computed, any ineffective portion of $e(k)$ is removed and saved to be added to the next error sample.
  - Apply the trapezoidal rule. The I-action term of position version becomes

$$f_I(k) = \sum_{i=0}^{k} \frac{1}{2}[e(i) + e(i-1)]K_I T \qquad (2.2\text{-}9)$$

- The output from D action in analog controllers is constant if the error signal increases at a constant rate. However, D action in digital controllers can produce a fluctuating output for such an error signal. The effect results from the round-off required by the finite word length of the machine. This behavior is referred to as derivative-mode kick.

- Solutions
  - Improve the resolution by increasing the word length of the computer.
  - Improve the approximation by using values of the sampled error signal at more instants. For example, in the velocity algorithm, the D-action term is replaced by the one obtained from a four-point central-difference technique. Let *m* be the mean of the previous four error samples.

$$m = \frac{e(k) + e(k-1) + e(k-2) + e(k-3)}{4} \qquad (2.2\text{-}10)$$

For $\hat{e}(k) = e(k) - m$, the new D-action term is

$$f_D(k) = \frac{K_D}{4T}\left[\frac{\hat{e}(k)}{1.5} + \frac{\hat{e}(k-1)}{0.5} + \frac{\hat{e}(k-2)}{0.5} + \frac{\hat{e}(k-3)}{1.5}\right] = \frac{K_D}{6T}[e(k) + 3e(k-1) - 3e(k-2) - e(k-3)] \qquad (2.2\text{-}11)$$

  - Another form of derivative kick occurs when the command input is a step function. The D action is the most sensitive to resulting rapid change in the error samples. This effect can be eliminated by reformulating the control algorithm as follows. To do this, I action must be included. The velocity algorithm for PID control.

$$f(k) = f(k-1) + k_p[e(k) - e(k-1)] + K_I Te(k) + \frac{K_D}{T}[e(k) - 2e(k-1) + e(k-2)] \qquad (2.2\text{-}12)$$

The error is $e(k) = r(k) - c(k)$, where *r* and *c* are the set point and output.

$$f(k) = f(k-1) + k_p[c(k-1) - c(k)] + K_I T[r - c(k)] + \frac{K_D}{T}[-c(k) + 2c(k-1) - c(k-2)] \qquad (2.2\text{-}13)$$

2.3 Pulse Transfer Functions for Digital Control Laws

The PID algorithm in time domain,

$$f(t) = K_p e(t) + K_I \int_0^t e dt + K_D \frac{de}{dt} \tag{2.3-1}$$

$$\frac{df}{dt} = K_p \frac{de}{dt} + K_I e + K_D \frac{d^2 e}{dt^2} \tag{2.3-2}$$

$$f(k) = f(k-1) + k_p [e(k) - e(k-1)] + K_I T e(k) + \frac{K_D}{T}[e(k) - 2e(k-1) + e(k-2)] \tag{2.3-3}$$

$$\frac{F(z)}{E(z)} = \frac{a_1 z^2 + a_2 z + a_3}{z(z-1)} \tag{2.3-4}$$

where $a_1 = K_p + K_I T + a_3$, $a_2 = -(K_p + 2a_3)$, and $a_3 = K_D / T$.

Controller Design Methods

Method 1: The controller design is done in the *s* domain, and the gain values K_p, K_I, K_D, are computed using the continuous-time methods. The resulting analog control law $G_c(s)$ must then be converted to discrete-time form with one of the approximation techniques.

Method 2: The performance specifications are given in terms of the desired continuous-time response and/or desired root locations in the *s* plane. From these the corresponding root locations in the *z* plane are found, and a discrete control law $G_c(z)$ is designed.

Method 3: The performance specifications are given in terms of the desired discrete-time response and/or desired root locations in the *z* plane. The rest of the procedure follows Method 2.

2.4 Direct Design of Digital Control Algorithms

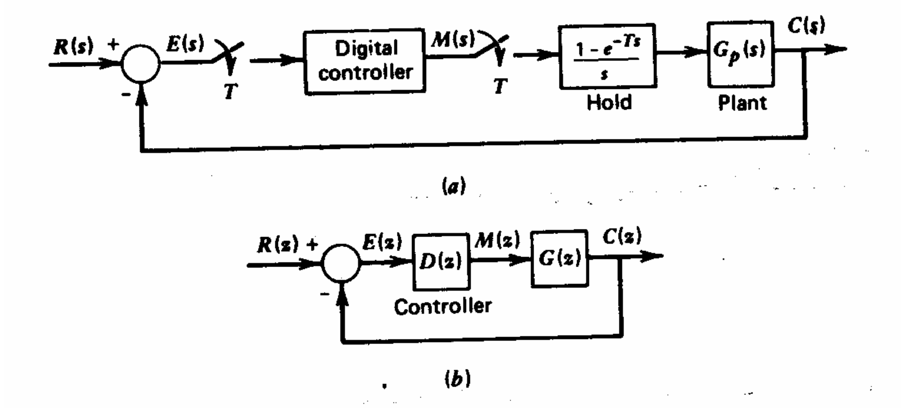


Figure 2.4-1 Digital Controller Configurations in the *s* Domain and the *z* Domain

$$T(z) = \frac{C(z)}{R(z)} = \frac{G(z)D(z)}{1 + G(z)D(z)} \tag{2.4-1}$$

$$D(z) = \frac{T(z)}{G(z)[1 - T(z)]} \tag{2.4-2}$$

$$T(z) = \frac{C(z)}{R(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots} \tag{2.4-3}$$

$$a_0 c(k) = -a_1 c(k-1) - a_2 c(k-2) - \cdots + b_0 r(k) + b_1 r(k-1) + b_2 r(k-2) + \cdots \tag{2.4-4}$$

2.5 Frequency Response Methods

If a sinusoid of circular frequency ω and amplitude *A* is sampled with a sampling period *T*, the resulting sequence is

$$u(k) = A \sin k\omega T \tag{2.5-1}$$

If this sequence is applied as an input to a stable system whose transfer function is *T*(*z*), the steady-state output is

$$y(k) = B \sin(k\omega T + \phi) \tag{2.5-2}$$

$$M = \frac{B}{A} = \left| T(e^{i\omega T}) \right| \tag{2.5-3}$$

$$\phi = \angle T(e^{i\omega T}) \tag{2.5-4}$$

Franklin and Powell have proposed the transformation

$$w = \frac{2}{T}\frac{z-1}{z+1} = \frac{2}{T}\frac{e^{sT}-1}{e^{sT}+1} = \frac{2}{T}\tanh\frac{sT}{2} \tag{2.5-5}$$

When $s = i\omega$, then $z = \exp(i\omega T)$.

$$w = i\frac{2}{T}\tan\frac{\omega T}{2} = iv \tag{2.5-6}$$

Given the open-loop transfer function *G*(*z*)*H*(*z*), substitute

$$z = \frac{1 + wT/2}{1 - wT/2} \tag{2.5-7}$$

The open-loop transfer function is now *G*(*w*)*H*(*w*), and the Bode design procedure is the same as before.