State-Space Control Theory

Modeling and analysis approaches of linear systems

- Transfer function or frequency-domain approach
- State-space approach
    - All the differential equations are first-order equations.
    - The number of first-order differential equations is equal to the order of the system.
    - The dynamic variables that appear in the system of first-order equations are called the state variables.
    - The number of state variables in the model of a physical process is unique, although the identity of these variables may not be unique

1 Physical Notion of System State

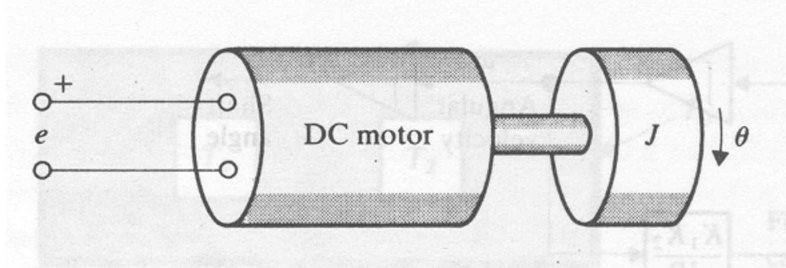


Figure 1-1 DC Motor Driving Inertial Load

$$\tau = K_1 i \tag{1-1}$$

$$v = K_2 \omega \tag{1-2}$$

When armature inductance is small and negligible,

$$e - v = Ri \tag{1-3}$$

When viscosity friction is small and negligible,

$$\tau = J \frac{d\omega}{dt} \tag{1-4}$$

$$J \frac{d\omega}{dt} = K_1 i = \frac{K_1}{R}(e - v) \tag{1-5}$$

$$J \frac{d\omega}{dt} = \frac{K_1}{R} e - \frac{K_1 K_2}{R} \omega \tag{1-6}$$

$$\frac{d\omega}{dt} = -\frac{K_1 K_2}{JR}\omega + \frac{K_1}{JR}e \tag{1-7}$$

$$\frac{d\theta}{dt} = \omega \tag{1-8}$$

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 0 & -K_1 K_2/(JR)\end{bmatrix}\begin{bmatrix}\theta \\ \omega\end{bmatrix} + \begin{bmatrix}0 \\ K_1/(JR)\end{bmatrix}e \tag{1-9}$$

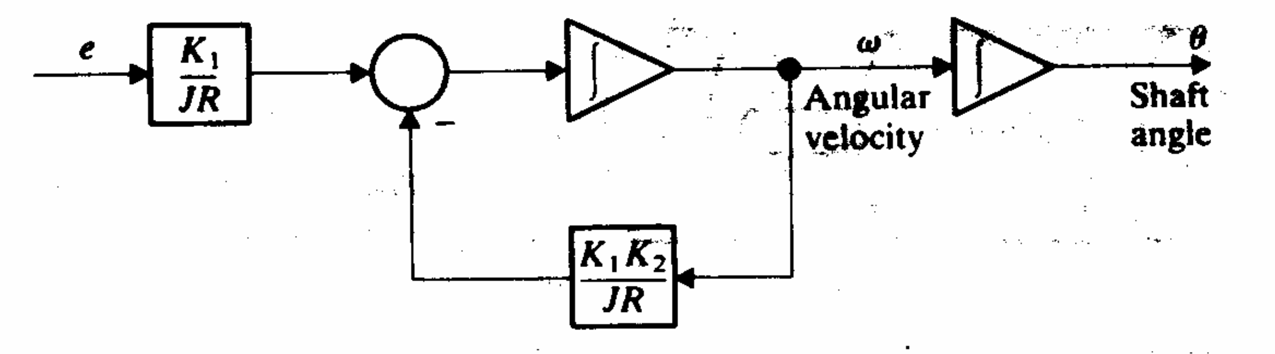


Figure 1-2 Block Diagram Representing DC Motor Driving Inertial Load

For k order system, state variables: x_1, x_2,…, x_k

Control inputs: u_1, u_2,…, u_l.

General form of the dynamic system

$$\dot{x}_1 = \frac{dx_1}{dt} = f_1(x_1, x_2, \cdots, x_k, u_1, u_2, \cdots, u_l, t)$$

$$\dot{x}_2 = \frac{dx_2}{dt} = f_2(x_1, x_2, \cdots, x_k, u_1, u_2, \cdots, u_l, t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (1\text{-}10)$$

$$\dot{x}_k = \frac{dx_k}{dt} = f_k(x_1, x_2, \cdots, x_k, u_1, u_2, \cdots, u_l, t)$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \text{ and } u = \begin{bmatrix} u_1 \\ \vdots \\ u_l \end{bmatrix} \qquad (1\text{-}11)$$

x: state vector

u: input vector

$$\dot{x} = \frac{dx}{dt} = f(x,u,t) \qquad (1\text{-}12)$$

In the state-space model of a linear process, the general differential equations take the special form:

$$\dot{x}_1 = \frac{dx_1}{dt} = a_{11}(t)x_1 + \cdots + a_{1k}(t)x_k + b_{11}(t)u_1 + \cdots + b_{1l}(t)u_l$$

$$\dot{x}_2 = \frac{dx_2}{dt} = a_{21}(t)x_1 + \cdots + a_{2k}(t)x_k + b_{21}(t)u_1 + \cdots + b_{2l}(t)u_l$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (1\text{-}13)$$

$$\dot{x}_k = \frac{dx_k}{dt} = a_{k1}(t)x_1 + \cdots + a_{kk}(t)x_k + b_{k1}(t)u_1 + \cdots + b_{kl}(t)u_l$$

State equation,

$$\dot{x} = \frac{dx}{dt} = A(t)x + B(t)u \quad (1\text{-}14)$$

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1k}(t) \\ \vdots & \ddots & \vdots \\ a_{k1}(t) & \cdots & a_{kk}(t) \end{bmatrix} \text{ and } B(t) = \begin{bmatrix} b_{11}(t) & \cdots & b_{1l}(t) \\ \vdots & \ddots & \vdots \\ b_{k1}(t) & \cdots & b_{kl}(t) \end{bmatrix} \quad (1\text{-}15)$$

$A(t)$: state matrix

$B(t)$: input matrix

For linear time-invariant processes, the state equation

$$\dot{x} = Ax + Bu \quad (1\text{-}16)$$

where A and B are constant matrices.

Measured outputs: $y_1, y_2, \ldots, y_m$.

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix} \tag{1-17}$$

$y(t)$: output vector or observation vector

In a linear system the output vector is also a linear combination of the state and the input, observation equation

$$y(t) = C(t)x(t) + D(t)u(t) \tag{1-18}$$

$C(t)$: output matrix

For time-invariant processes, $C(t)$ and $D(t)$ are constant matrices.

Two types of input:

1. Control inputs, u, produced intentionally by the operation of the control system
2. Exogenous inputs, x_0, present in the environment and not subject to control within the system

The general representation of a linear system

$$\dot{x} = Ax + Bu + Ex_0 \tag{1-19}$$

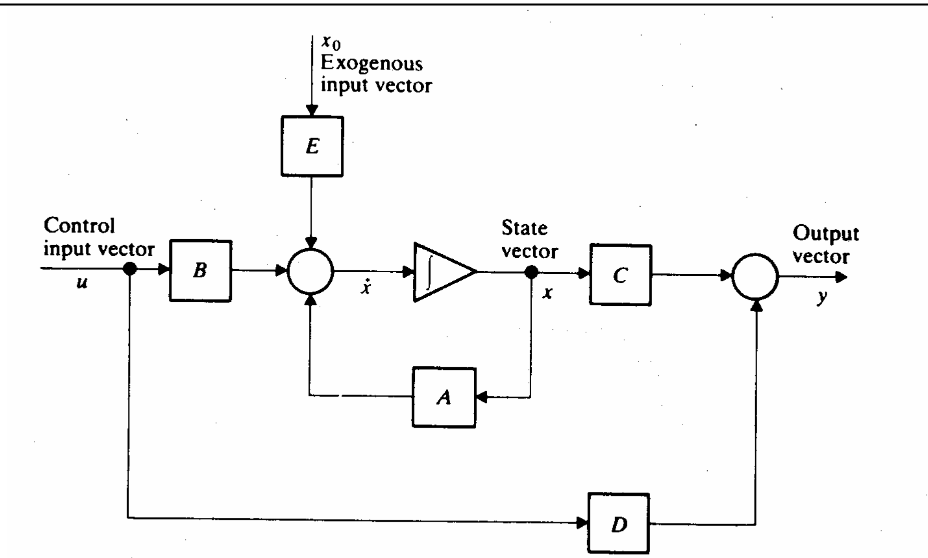


Figure 1-3 Block-Diagram Representation of General Linear System

Consider the system defined by

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u \tag{1}$$

Let's choose the state variables as

$$x_1 = y \tag{2}$$

$$x_2 = \dot{y} \tag{3}$$

$$x_3 = \ddot{y} \tag{4}$$

Then we obtain

$$\dot{x}_1 = x_2 \tag{5}$$

$$\dot{x}_2 = x_3 \tag{6}$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 + 6u \tag{7}$$

Or in the matrix form of state-space representation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u \tag{8}$$

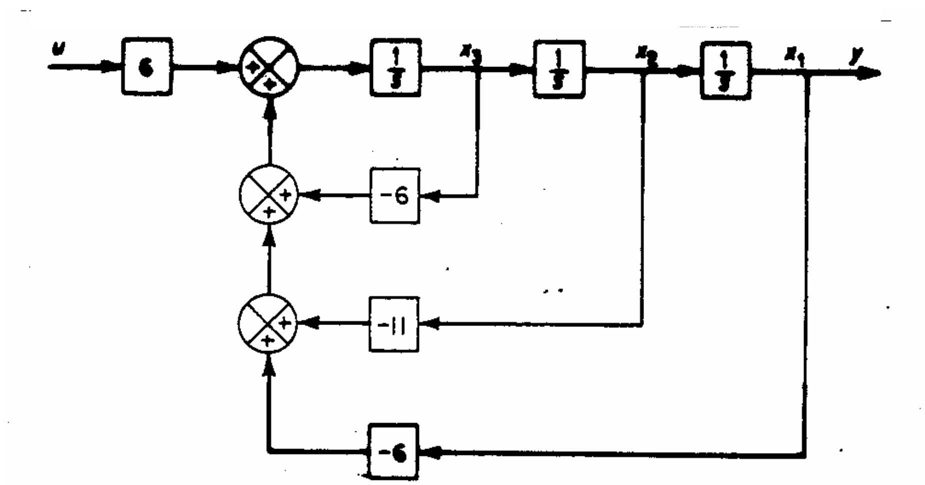
and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{9}$$

or

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{10}$$

Block diagram, representing this system, is shown below.



Consider again the system

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u \tag{11}$$

Taking Laplace transformation of the system with zero initial condition, we get

$$s^3 y(s) + 6s^2 y(s) + 11sy(s) + 6y(s) = 6u(s) \tag{12}$$

The system transfer function then becomes

$$\frac{y(s)}{u(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6} = \frac{6}{(s+1)(s+2)(s+3)} \tag{13}$$

By expanding the transfer function into partial fractions, we obtain

$$\frac{y(s)}{u(s)} = \frac{3}{s+1} - \frac{6}{s+2} + \frac{3}{s+3} \tag{14}$$

Hence,

$$y(s) = \frac{3}{s+1}u(s) - \frac{6}{s+2}u(s) + \frac{3}{s+3}u(s) \tag{15}$$

Let's define

$$x_1(s) = \frac{3}{s+1}u(s) \text{ or } (s+1)x_1(s) = 3u(s) \tag{16}$$

$$x_2(s) = -\frac{6}{s+2}u(s) \text{ or } (s+2)x_2(s) = -6u(s) \tag{17}$$

$$x_3(s) = \frac{3}{s+3}u(s) \text{ or } (s+3)x_3(s) = 3u(s) \tag{18}$$

The inverse Laplace transformations of (16)-(18) give

$$\dot{x}_1 = -x_1 + 3u \tag{19}$$

$$\dot{x}_2 = -2x_2 - 6u \tag{20}$$

$$\dot{x}_3 = -3x_3 + 3u \tag{21}$$

Or in the matrix form of state-space representation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} u \tag{22}$$

Since from (15)-(18),

$$y(s) = x_1(s) + x_2(s) + x_3(s) \tag{23}$$

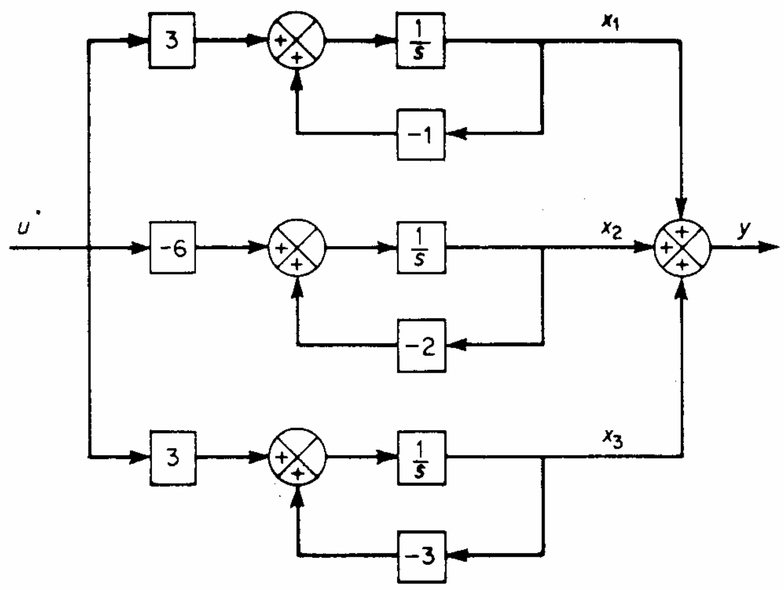
or

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{24}$$

Thus

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \tag{25}$$

Block diagram, representing this system, is shown below.



2 Solution of Linear Differential Equations in State-Space Form

General differential equation of an unforced system

$$\dot{x} = Ax \tag{2-1}$$

The solution is the form of

$$x(t) = e^{At}c \tag{2-2}$$

where e^{At} is the matrix exponential function

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,\ e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = 2.7183;\ e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots \tag{2-3}$$

At time τ, the state $x(\tau)$ is given.

$$x(\tau) = e^{A\tau}c \tag{2-4}$$

$$c = (e^{A\tau})^{-1}x(\tau) \tag{2-5}$$

The general solution

$$x(t) = e^{At}(e^{A\tau})^{-1}x(\tau) = e^{A(t-\tau)}x(\tau) \tag{2-6}$$

General differential equation of a forced system,

$$\dot{x} = Ax + Bu \tag{2-7}$$

The solution is the form of

$$x(t) = e^{At}c(t) \tag{2-8}$$

$$Ae^{At}c(t) + e^{At}\dot{c}(t) = Ae^{At}c(t) + Bu(t) \tag{2-9}$$

$$\dot{c}(t) = e^{-At}Bu(t) \tag{2-10}$$

$$c(t) = \int_T^t e^{-A\lambda}Bu(\lambda)d\lambda \tag{2-11}$$

$$x(t) = e^{At}\int_T^t e^{-A\lambda}Bu(\lambda)d\lambda = \int_T^t e^{A(t-\lambda)}Bu(\lambda)d\lambda \tag{2-12}$$

The combined solution of free and forced response

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_T^t e^{A(t-\lambda)}Bu(\lambda)d\lambda \tag{2-13}$$

At time τ, the state $x(\tau)$ is given.

$$x(\tau) = x(\tau) + \int_T^\tau e^{A(t-\lambda)}Bu(\lambda)d\lambda \tag{2-14}$$

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_\tau^t e^{A(t-\lambda)}Bu(\lambda)d\lambda \tag{2-15}$$

$$y = Cx \tag{2-16}$$

$$y(t) = Ce^{A(t-\tau)}x(\tau) + \int_\tau^t Ce^{A(t-\lambda)}Bu(\lambda)d\lambda \tag{2-17}$$

When B and C are time-varying, the solution is generalized to

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_{\tau}^{t} e^{A(t-\lambda)}B(\lambda)u(\lambda)d\lambda \tag{2-18}$$

$$y(t) = C(t)e^{A(t-\tau)}x(\tau) + \int_{\tau}^{t} C(t)e^{A(t-\lambda)}B(\lambda)u(\lambda)d\lambda \tag{2-19}$$

When A is time-varying, the solution of $\dot{x} = A(t)x$ is generalized to

$$x(t) = \phi(t,\tau)x(\tau) \tag{2-20}$$

The matrix $\phi(t, \tau)$ that relates the state at time t to the state at time τ is generally known as the state-transition matrix.

The complete general solution

$$x(t) = \phi(t,\tau)x(\tau) + \int_{\tau}^{t} \phi(t,\lambda)B(\lambda)u(\lambda)d\lambda \tag{2-21}$$

$$y(t) = C(t)\phi(t,\tau)x(\tau) + \int_{\tau}^{t} C(t)\phi(t,\lambda)B(\lambda)u(\lambda)d\lambda \tag{2-22}$$

The differential equation for the position of a mass without friction to which an external force *f* is applied is

$$\ddot{x} = f/m = u \tag{1}$$

Defining the state variables by

$$x_1 = x, x_2 = \dot{x} \tag{2}$$

result in the state-space form

$$\dot{x}_1 = x_2 \tag{3}$$

$$\dot{x}_2 = u \tag{4}$$

Thus,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5}$$

Using the series definition

$$e^{At} = I + At + A^2t^2/2 + A^3t^3/3! + \cdots \tag{6}$$

we obtain the state transition matrix

$$\phi(t) = e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \tag{7}$$

Thus, the solution which follows the general solution

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_{\tau}^{t} e^{A(t-\lambda)}B(\lambda)u(\lambda)d\lambda \tag{8}$$

is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} 1 & t-\lambda \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}u(\lambda)d\lambda = \begin{bmatrix} x_1(0) + tx_2(0) + \int_0^t (t-\lambda)u(\lambda)d\lambda \\ x_2(0) + \int_0^t u(\lambda)d\lambda \end{bmatrix} \tag{9}$$

3 Solution by the Laplace Transform: The Resolvent

$$L[f(t)] = f(s) = \int_0^{\infty} f(t)e^{-st}dt \tag{3-1}$$

Laplace transform can be applied also when $f(t)$ is a vector.

$$L[f(t)] = L\begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix} = \begin{bmatrix} L[f_1(t)] \\ \vdots \\ L[f_n(t)] \end{bmatrix} = \begin{bmatrix} f_1(s) \\ \vdots \\ f_n(s) \end{bmatrix} = f(s) \tag{3-2}$$

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3-3}$$

$$sx(s) - x(0) = Ax(s) + Bu(s) \tag{3-4}$$

$$(sI - A)x(s) = x(0) + Bu(s) \tag{3-5}$$

$$x(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}Bu(s) \tag{3-6}$$

$$L[e^{at}] = \frac{1}{s-a} = (s-a)^{-1} \tag{3-7}$$

$$L[e^{At}] = (sI - A)^{-1} \tag{3-8}$$

$$L\left[\int_0^t f(t-\lambda)g(\lambda)d\lambda\right] = f(s)g(s) \tag{3-9}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\lambda)}Bu(\lambda)d\lambda \tag{3-10}$$

e^{At}: state transition matrix for a time-invariant system

$$\phi(s) = (sI - A)^{-1} \tag{3-11}$$

ϕ : resolvent of A

- The steps in calculating the state-transition matrix using the resolvent:

1. Calculate sI-A.
2. Obtain the resolvent by inverting $(sI\text{-}A)^{-1}$.
3. Obtain the state-transition matrix by taking the inverse Laplace transform of the resolvent, element by element.

   For a general kth-order system the matrix sI-A has the following appearance

$$sI - A = \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1k} \\ -a_{21} & s - a_{22} & \cdots & -a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{k1} & -a_{k2} & \cdots & s - a_{kk} \end{bmatrix} \tag{3-12}$$

$$(sI - A)^{-1} = \frac{adj(sI - A)}{|\, sI - A\,|} \tag{3-13}$$

$$|\, sI - A\,| = s^k + a_1 s^{k-1} + \cdots + a_k \tag{3-14}$$

- $|sI\text{-}A\,|$: characteristic polynomial of the matrix A.
- The roots of characteristic polynomial are called the characteristic roots or the eigenvalues, or the poles of the system and determine the essential features of the unforced dynamic behavior of the system.

$$adj(sI - A) = E_1 s^{k-1} + E_2 s^{k-2} + \cdots + E_k \quad (3\text{-}15)$$

$$(sI - A)^{-1} = \frac{E_1 s^{k-1} + \cdots E_k}{s^k + a_1 s^{k-1} + \cdots a_k} \quad (3\text{-}16)$$

$$|sI - A| I = (sI - A)(E_1 s^{k-1} + E_2 s^{k-2} + \cdots + E_k) \quad (3\text{-}17)$$

$$s^k I + a_1 s^{k-1} I + \cdots + a_k I = s^k E_1 + s^{k-1}(E_2 - AE_1) + \cdots + s(E_k - AE_{k-1}) - AE_k \quad (3\text{-}18)$$

$$\begin{aligned} E_1 &= I \\ E_2 - AE_1 &= a_1 I \\ E_3 - AE_2 &= a_2 I \\ &\cdots\cdots\cdots \\ E_k - AE_{k-1} &= a_{k-1} I \\ -AE_k &= a_k I \end{aligned} \quad (3\text{-}19)$$

The subsequent coefficients

$$\begin{aligned} E_2 &= AE_1 + a_1 I \\ E_3 &= AE_2 + a_2 I \\ &\cdots\cdots\cdots \\ E_k &= AE_{k-1} + a_{k-1} I \end{aligned} \quad (3\text{-}20)$$

$$E_{k+1} = AE_k + a_k I = 0 \quad (3\text{-}21)$$

The dynamic of a dc motor driving an inertial load is represented by

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 0 & -K_1K_2/(JR)\end{bmatrix}\begin{bmatrix}\theta \\ \omega\end{bmatrix} + \begin{bmatrix}0 \\ K_1/(JR)\end{bmatrix}e \tag{1}$$

(1) is equivalent with

$$\dot{x} = Ax + Bu \tag{2}$$

The matrices of the state-space characterization are

$$A = \begin{bmatrix}0 & 1 \\ 0 & -\alpha\end{bmatrix}, B = \begin{bmatrix}0 \\ \beta\end{bmatrix} \tag{3}$$

when

$$\alpha = \frac{K_1K_2}{JR} \text{ and } \beta = \frac{K_1}{JR} \tag{4}$$

Thus the resolvent is

$$\phi(s) = (sI - A)^{-1} = \begin{bmatrix}s & -1 \\ 0 & s+\alpha\end{bmatrix}^{-1} = \frac{1}{s(s+\alpha)}\begin{bmatrix}s+\alpha & 1 \\ 0 & s\end{bmatrix} = \begin{bmatrix}1/s & 1/(s(s+\alpha)) \\ 0 & 1/(s+\alpha)\end{bmatrix} \tag{5}$$

Finally, taking the inverse Laplace transforms of each term in (5) we obtain

$$e^{At} = \phi(t) = \begin{bmatrix}1 & (1-e^{-\alpha t})/\alpha \\ 0 & e^{-\alpha t}\end{bmatrix} \tag{6}$$

4 Input-Output Relations: Transfer Functions

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4-1}$$

Initial state is not considered in transfer function determination.

$$x(s) = (sI - A)^{-1} Bu(s) \tag{4-2}$$

$$y(t) = Cx(t) + Du(t) \tag{4-3}$$

$$y(s) = Cx(s) + Du(s) \tag{4-4}$$

Transfer-function matrix

$$H(s) = C(sI - A)^{-1} B + D \tag{4-5}$$

The corresponding impulse-response matrix

$$H(t) = Ce^{At} B + D\delta(t) \tag{4-6}$$

In case that there is no direct connection from the input to the output, $D = 0$, the degree of the numerator in $H(s)$ is always lower than the degree of the denominator.

$$H(s) = C(sI - A)^{-1} B = \frac{C[E_1 s^{k-1} + E_2 s^{k-2} + \cdots + E_k]B}{|sI - A|}$$

$$= \frac{CBs^{k-1} + CE_2 Bs^{k-2} + \cdots + CE_k B}{s^k + a_1 s^{k-1} + \cdots + a_k} \tag{4-7}$$

5 Transformation of State Variables

A linear transformation between 2 formulations of state variables

$$z = Tx \quad (5\text{-}1)$$

z: state vector in the new formulation

x: state vector in the original formulation.

When the transformation matrix T is a nonsingular k by k matrix.

$$x = T^{-1}z \quad (5\text{-}2)$$

$$\dot{x} = Ax + Bu \quad (5\text{-}3)$$

$$y = Cx + Du \quad (5\text{-}4)$$

$$T^{-1}\dot{z} = AT^{-1}z + Bu \quad (5\text{-}5)$$

$$\dot{z} = TAT^{-1}z + TBu \quad (5\text{-}6)$$

$$y = CT^{-1}z + Du \quad (5\text{-}7)$$

$$\dot{z} = \overline{A}z + \overline{B}u \quad (5\text{-}8)$$

$$y = \overline{C}z + \overline{D}u \quad (5\text{-}9)$$

$$\overline{A} = TAT^{-1},\ \overline{B} = TB,\ \overline{C} = CT^{-1},\ \overline{D} = D \quad (5\text{-}10)$$

- All formulations of the same system always have the same characteristic polynomial.

Consider again the system defined by

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u \qquad (1)$$

At least the following two pairs are choices to represent this system.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad (2)$$

and

$$A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \qquad (3)$$

The transformation matrix below is used to transform the system from (3) into (2),

$$T = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \qquad (4)$$

$$TA_2T^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = A_1 \qquad (5)$$

$$TB_2 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = B_1 \quad (6)$$

$$C_2 T^{-1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = C_1 \quad (7)$$

$$D_2 = [0] = D_1 \quad (8)$$

- To determine transformation matrix, T, when A and $\overline{A}$ are given. Solve k^2 unknown in T in

$$\overline{A} = TAT^{-1},\ \overline{B} = TB,\ \overline{C} = CT^{-1},\ \overline{D} = D \quad (5\text{-}11)$$

6 State-Space Representation of Transfer Functions: Canonical Forms

6.1 First Companion Form

- In the first companion form, the coefficients of the denominator of the transfer function appear in one of the row of the A matrix.

Transfer function of a single-input, single-output system of the form

$$H(s) = \frac{y(s)}{u(s)} = \frac{1}{s^k + a_1 s^{k-1} + \cdots + a_k} \tag{6.1-1}$$

$$(s^k + a_1 s^{k-1} + \cdots + a_k)\,y(s) = u(s) \tag{6.1-2}$$

$$D^k y + a_1 D^{k-1} y + \cdots + a_k y = u \tag{6.1-3}$$

where $D^i y$: $d^i y/dt^i$.

$$D^k y = -a_1 D^{k-1} y - \cdots - a_k y + u \tag{6.1-4}$$

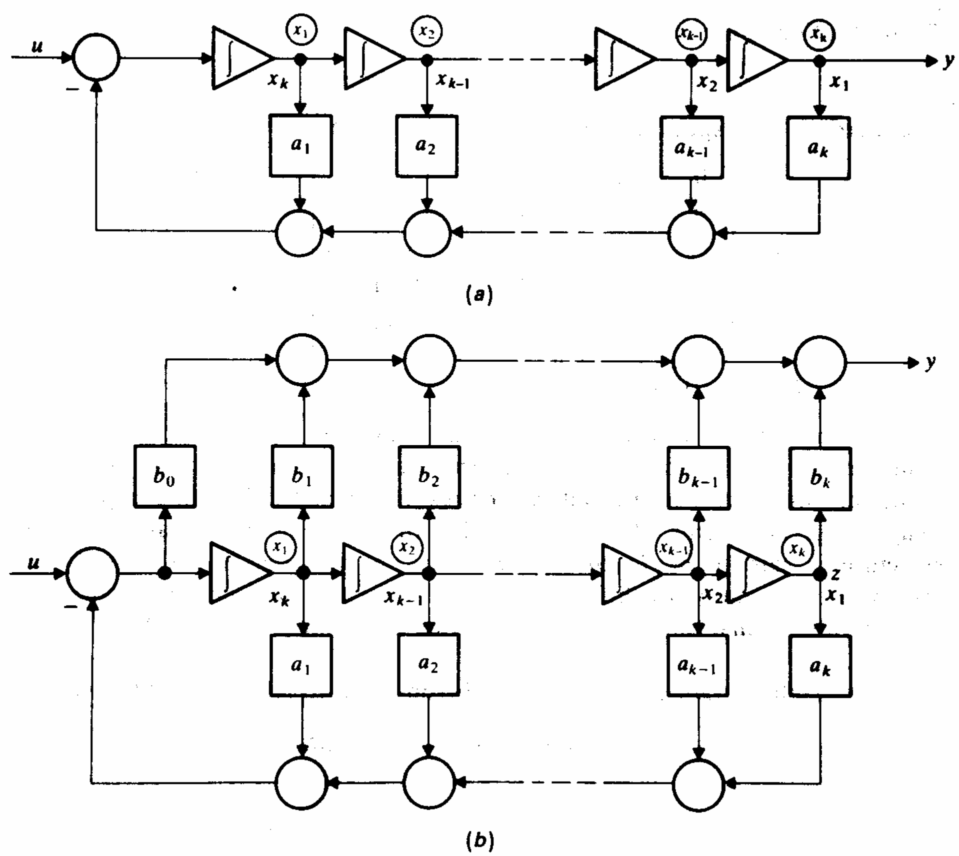


Figure 6.1-1 State-Space Realization of Transfer Functions in First Companion Form

(a) $H(s) = \dfrac{1}{s^k + a_1 s^{k-1} + \cdots + a_k}$, (b) $H(s) = \dfrac{b_0 s^k + b_1 s^{k-1} + \cdots + b_k}{s^k + a_1 s^{k-1} + \cdots + a_k}$

The corresponding state equations

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\cdots\ \cdots \\
\dot{x}_{k-1} &= x_k \\
\dot{x}_k &= -a_k x_1 - a_{k-1} x_2 - \cdots - a_1 x_k + u
\end{aligned}
\tag{6.1-5}
$$

The output equation

$$y = x_1 \tag{6.1-6}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 \end{bmatrix} \tag{6.1-7}$$

- The form of matrix A is said to be in ***companion form***.
- The different numbering of the state variables will make different form of matrices A, B, and C. All of them are valid

General transfer function

$$H(s) = \frac{y(s)}{u(s)} = \frac{b_0 s^k + b_1 s^{k-1} + \cdots + b_k}{s^k + a_1 s^{k-1} + \cdots + a_k} \tag{6.1-8}$$

$$\frac{y(s)}{u(s)} = \frac{y(s)}{z(s)} \cdot \frac{z(s)}{u(s)} = \frac{b_0 s^k + b_1 s^{k-1} + \cdots + b_k}{s^k + a_1 s^{k-1} + \cdots + a_k} \tag{6.1-9}$$

$$\frac{y(s)}{z(s)} = b_0 s^k + b_1 s^{k-1} + \cdots + b_k \tag{6.1-10}$$

$$\frac{z(s)}{u(s)} = \frac{1}{s^k + a_1 s^{k-1} + \cdots + a_k} \tag{6.1-11}$$

$$y(s) = (b_0 s^k + b_1 s^{k-1} + \cdots + b_k) z(s) \tag{6.1-12}$$

$$y = b_0 D^k z + b_1 D^{k-1} z + \cdots + b_k z \tag{6.1-13}$$

$$\begin{aligned} x_1 &= z \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\dots \dots \\ \dot{x}_{k-1} &= x_k \\ \dot{x}_k &= -a_k x_1 - a_{k-1} x_2 - \cdots - a_1 x_k + u \end{aligned} \tag{6.1-14}$$

$$y = (b_k - a_k b_0)x_1 + (b_{k-1} - a_{k-1}b_0)x_2 + \cdots + (b_1 - a_1 b_0)x_k + b_0 u \qquad (6.1\text{-}15)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ C = [b_k - a_k b_0 \quad b_{k-1} - a_{k-1}b_0 \quad \ldots \quad b_1 - a_1 b_0],\ D = [b_0] \qquad (6.1\text{-}16)$$

For single-input, multiple-output system

$$\frac{y_1(s)}{u(s)} = \frac{b_{01}s^k + b_{11}s^{k-1} + \cdots + b_{k1}}{s^k + a_1 s^{k-1} + \cdots + a_k}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (6.1\text{-}17)$$

$$\frac{y_l(s)}{u(s)} = \frac{b_{0l}s^k + b_{1l}s^{k-1} + \cdots + b_{kl}}{s^k + a_1 s^{k-1} + \cdots + a_k}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} b_{k1} - a_k b_{01} & b_{k-1,1} - a_{k-1}b_{01} & \cdots & b_{11} - a_1 b_{01} \\ b_{k2} - a_k b_{02} & b_{k-1,2} - a_{k-1}b_{02} & \cdots & b_{12} - a_1 b_{02} \\ \vdots & \vdots & \ddots & \vdots \\ b_{kl} - a_k b_{0l} & b_{k-1,l} - a_{k-1}b_{0l} & \cdots & b_{1l} - a_1 b_{0l} \end{bmatrix} \text{ and } D = \begin{bmatrix} b_{01} \\ b_{02} \\ \vdots \\ b_{0l} \end{bmatrix} \qquad (6.1\text{-}18)$$

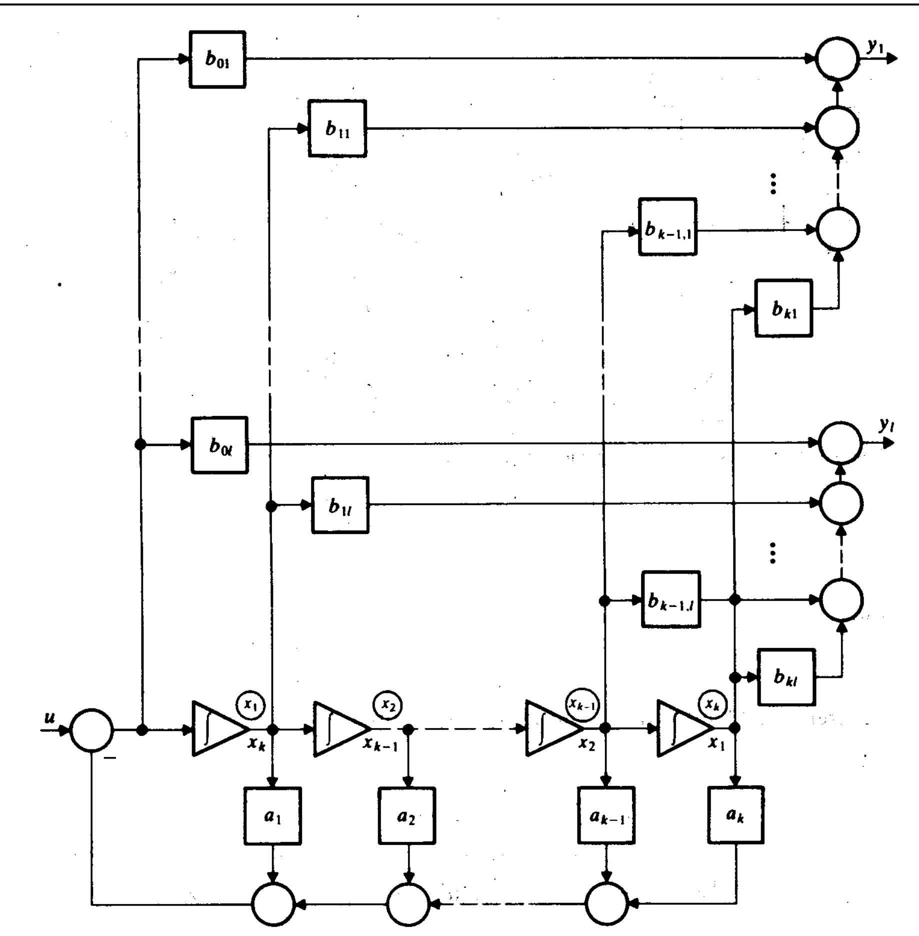


Figure 6.1-2 State-Space Realization of Single-Input, Multiple-Output System

When state variables are numbered from left to right instead of right to left.

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},\ C = [b_1 - a_1 b_0 \quad b_2 - a_2 b_0 \quad \ldots \quad b_{k-1} - a_{k-1} b_0 \quad b_k - a_k b_0],\ D = [b_0] \qquad (6.1\text{-}19)$$

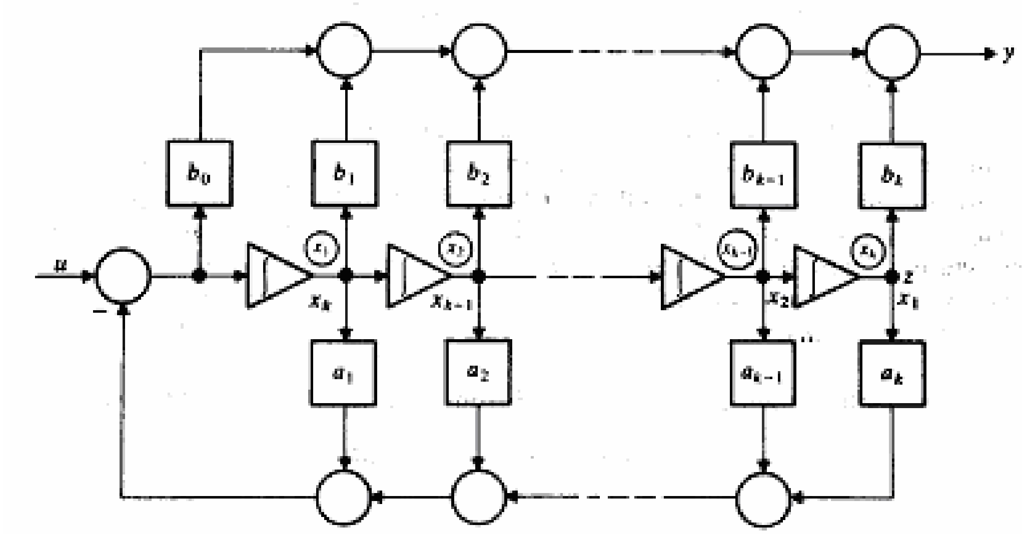


Figure 6.1-3 Numbering State Variables from Left to Right

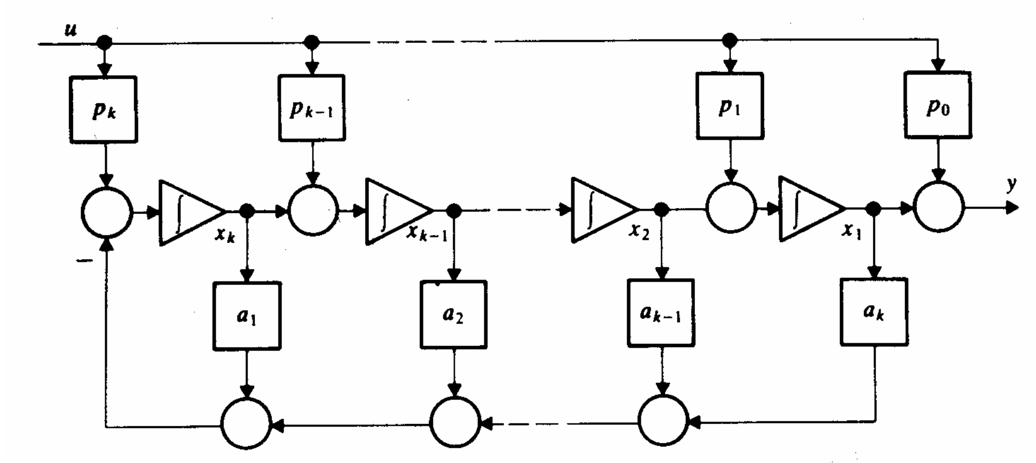


Figure 6.1-4 Alternative First Companion Form of State-Space Realization

$$\begin{aligned}
\dot{x}_1 &= x_2 + p_1 u \\
\dot{x}_2 &= x_3 + p_2 u \\
&\dots\dots\dots\dots \\
\dot{x}_{k-1} &= x_k + p_{k-1} u \\
\dot{x}_k &= -a_1 x_k - \cdots - a_k x_1 + p_k u
\end{aligned} \qquad (6.1\text{-}20)$$

$$y = x_1 + p_0 u \qquad (6.1\text{-}21)$$

$$
\begin{aligned}
Dy &= x_2 + p_1 u + p_0 Du \\
D^2 y &= x_3 + p_2 u + p_1 Du + p_0 D^2 u \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
D^{k-1} y &= x_k + p_{k-1} u + p_{k-2} Du + \cdots + p_1 D^{k-2} u + p_0 D^{k-1} u \\
D^k y &= -a_1 x_k - a_2 x_{k-1} - \cdots - a_k x_1 + p_k u + p_{k-1} Du + \cdots + p_1 D^{k-1} u + p_0 D^k u
\end{aligned}
\qquad (6.1\text{-}22)
$$

$$
\begin{aligned}
D^k y + a_1 D^{k-1} y + \cdots + a_{k-1} Dy + a_k y = {} & (p_k + a_1 p_{k-1} + \cdots + a_{k-1} p_1 + a_k p_0) u \\
& + (p_{k-1} + \cdots + a_{k-2} p_1 + a_{k-1} p_0) Du \\
& + \cdots + (p_1 + a_1 p_0) D^{k-1} u \\
& + (p_0) D^k u
\end{aligned}
\qquad (6.1\text{-}23)
$$

$$
H(s) = \frac{y(s)}{u(s)} = \frac{b_0 s^k + b_1 s^{k-1} + \cdots + b_k}{s^k + a_1 s^{k-1} + \cdots + a_k} \qquad (6.1\text{-}24)
$$

$$
D^k y + a_1 D^{k-1} y + \cdots + a_{k-1} Dy + a_k y = b_0 D^k u + b_1 D^{k-1} u + \cdots + b_k u \qquad (6.1\text{-}25)
$$

$$
\begin{aligned}
p_0 &= b_0 \\
p_1 + a_1 p_0 &= b_1 \\
&\cdots\cdots\cdots\cdots \\
p_{k-1} + \cdots + a_{k-2} p_1 + a_{k-1} p_0 &= b_{k-1} \\
p_k + a_1 p_{k-1} + \cdots + a_{k-1} p_1 + a_k p_0 &= b_k
\end{aligned}
\qquad (6.1\text{-}26)
$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_k & a_{k-1} & a_{k-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_k \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \qquad (6.1\text{-}27)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_1 \end{bmatrix},\ B = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{k-1} \\ p_k \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix},\ D = [p_0] \qquad (6.1\text{-}28)$$

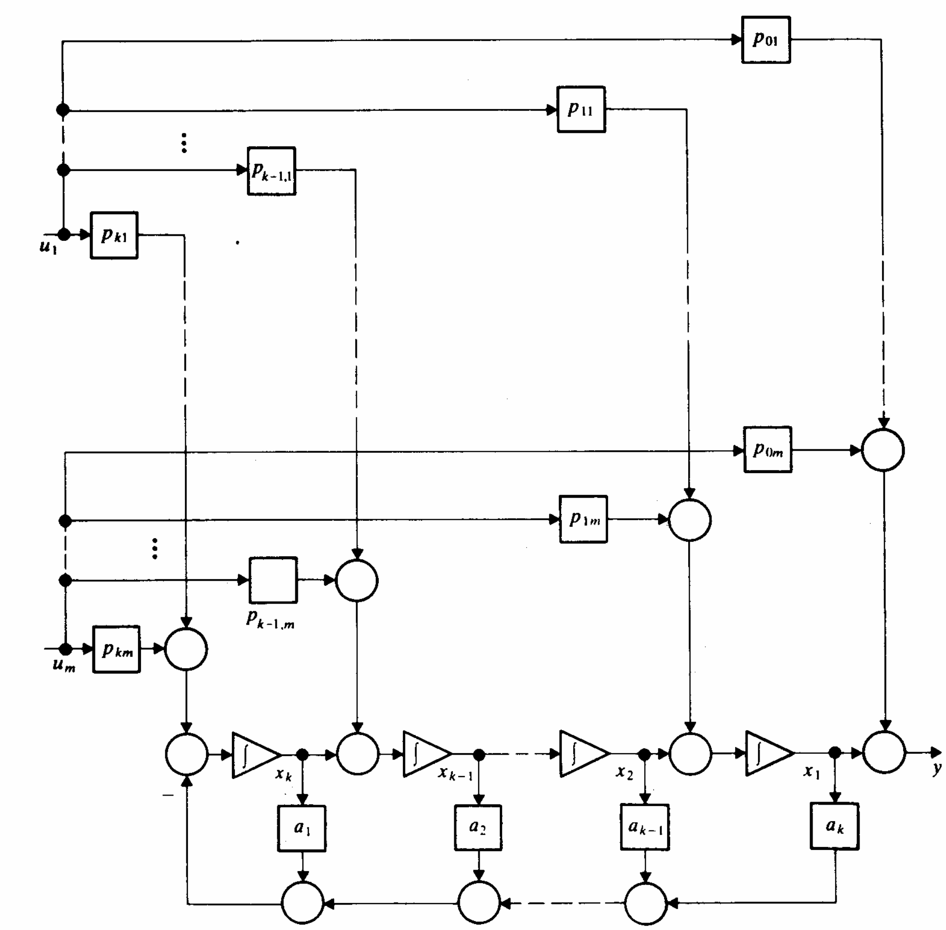


Figure 6.1-5 State-Space Realization of Multiple-Input, Single-Output System

6.2 Second Companion Form

- In the first companion form, the coefficients of the denominator of the transfer function appear in one of the column of the A matrix.

For a single-input, single-output system,

$$H(s) = \frac{y(s)}{u(s)} = \frac{b_0 s^k + b_1 s^{k-1} + \cdots + b_k}{s^k + a_1 s^{k-1} + \cdots + a_k} \tag{6.2-1}$$

$$(s^k + a_1 s^{k-1} + \cdots + a_k) y(s) = (b_0 s^k + b_1 s^{k-1} + \cdots + b_k) u(s) \tag{6.2-2}$$

$$s^k [y(s) - b_0 u(s)] + s^{k-1} [a_1 y(s) - b_1 u(s)] + \cdots + [a_k y(s) - b_k u(s)] = 0 \tag{6.2-3}$$

$$y(s) = b_0 u(s) + \frac{1}{s}[b_1 u(s) - a_1 y(s)] + \cdots + \frac{1}{s^k}[b_k u(s) - a_k y(s)] \tag{6.2-4}$$

$$x_i(s) = \frac{1}{s}[b_i u(s) - a_i y(s)] + \frac{1}{s} x_{i+1} \tag{6.2-5}$$

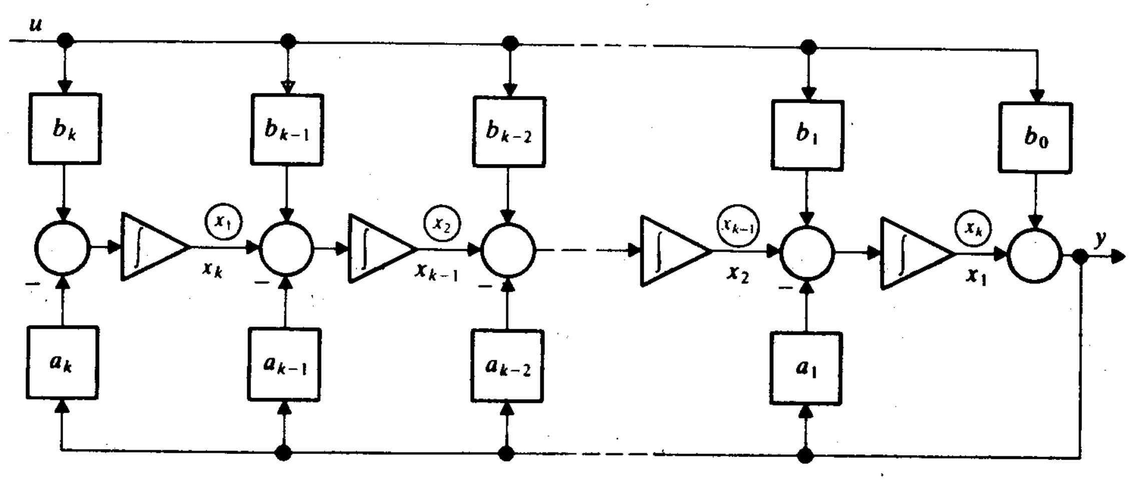


Figure 6.2-1 State-Space Realization of Single-Input, Single-Output System

$$\dot{x}_1 = x_2 - a_1(x_1 + b_0 u) + b_1 u$$
$$\dot{x}_2 = x_3 - a_2(x_1 + b_0 u) + b_2 u$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (6.2\text{-}6)$$
$$\dot{x}_{k-1} = x_k - a_{k-1}(x_1 + b_0 u) + b_{k-1} u$$
$$\dot{x}_k = -a_k(x_1 + b_0 u) + b_k u$$

$$y = x_1 + b_0 u \qquad (6.2\text{-}7)$$

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_k & 0 & 0 & \cdots & 0 \end{bmatrix},\ B = \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_k - a_k b_0 \end{bmatrix},\ C = [1\ 0\ 0\ \ldots\ 0],\ D = [b_0] \qquad (6.2\text{-}8)$$

6.3 Jordan Form: Partial Fraction Expansion

- This canonical form follows directly from the partial fraction expansion of the transfer functions.

When the poles of the transfer function are all different. The partial fraction expansion of the transfer function

$$H(s) = \frac{y(s)}{u(s)} = b_0 + \frac{r_1}{s - s_1} + \frac{r_2}{s - s_2} + \cdots + \frac{r_k}{s - s_k} \tag{6.3-1}$$

$$y = b_0 u + r_1 x_1 + r_2 x_2 + \cdots + r_k x_k \tag{6.3-2}$$

$$x_i = \frac{u}{s - s_i} \tag{6.3-3}$$

$$\dot{x}_i - s_i x_i = u \tag{6.3-4}$$

$$\dot{x}_i = s_i x_i + u \tag{6.3-5}$$

$$\begin{aligned} \dot{x}_1 &= s_1 x_1 + u \\ \dot{x}_2 &= s_2 x_2 + u \\ &\cdots\cdots\cdots\cdots \\ \dot{x}_k &= s_k x_k + u \end{aligned} \tag{6.3-6}$$

$$A = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_k \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \; C = [r_1 \;\; r_2 \;\; \ldots \;\; r_k], \; D = [b_0] \tag{6.3-7}$$

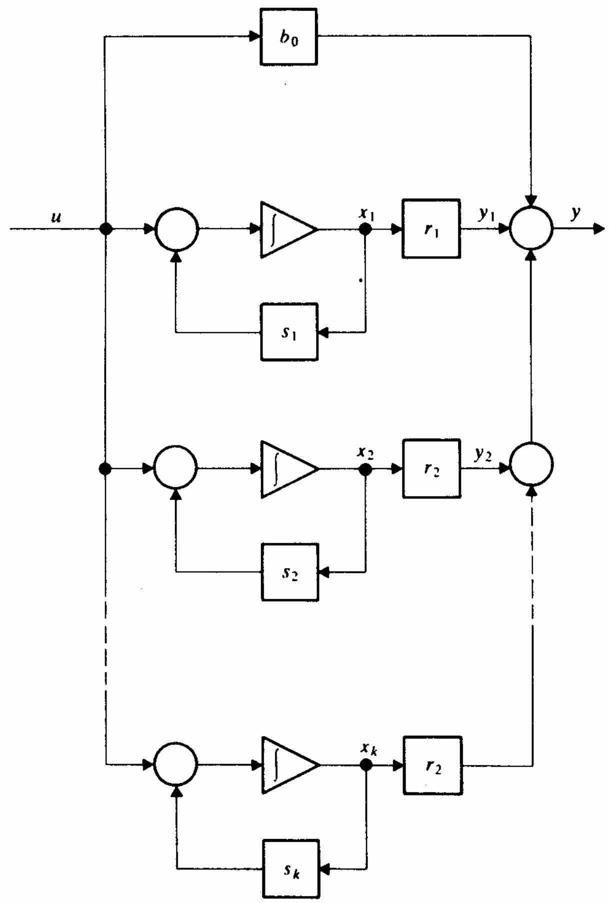


Figure 6.3-1 Complex Jordan Form of Transfer Function with Distinct Roots

When the poles are complex conjugates, $s_1 = -\sigma + j\omega$ and $s_2 = -\sigma - j\omega$, with corresponding residues $r_1 = \lambda + j\gamma$ and $r_2 = \lambda - j\gamma$.

$$H_{1,2} = \frac{y_{1,2}(s)}{u(s)} = \frac{\lambda + j\gamma}{s + \sigma - j\omega} + \frac{\lambda - j\gamma}{s + \sigma + j\omega} = \frac{2[\lambda s + (\lambda\sigma - \omega\gamma)]}{s^2 + 2\sigma s + \sigma^2 + \omega^2} \tag{6.3-8}$$

$$x_1 = \frac{u}{s^2 + 2\sigma s + \sigma^2 + \omega^2} \tag{6.3-9}$$

$$x_2 = \frac{us}{s^2 + 2\sigma s + \sigma^2 + \omega^2} \tag{6.3-10}$$

$$\dot{x}_1 = x_2 \tag{6.3-11}$$

$$\dot{x}_2 \rightarrow sx_2 = \frac{us^2}{s^2 + 2\sigma s + \sigma^2 + \omega^2} = -(\sigma^2 + \omega^2)x_1 - 2\sigma x_2 + u \tag{6.3-12}$$

$$y_{1,2} = 2(\lambda\sigma - \omega\gamma)x_1 + 2\lambda x_2 \tag{6.3-13}$$

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\sigma^2 + \omega^2) & -2\sigma \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{6.3-14}$$

$$y_{1,2} = [2(\lambda\sigma \text{-} \omega\gamma) \;\; 2\lambda] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \tag{6.3-15}$$

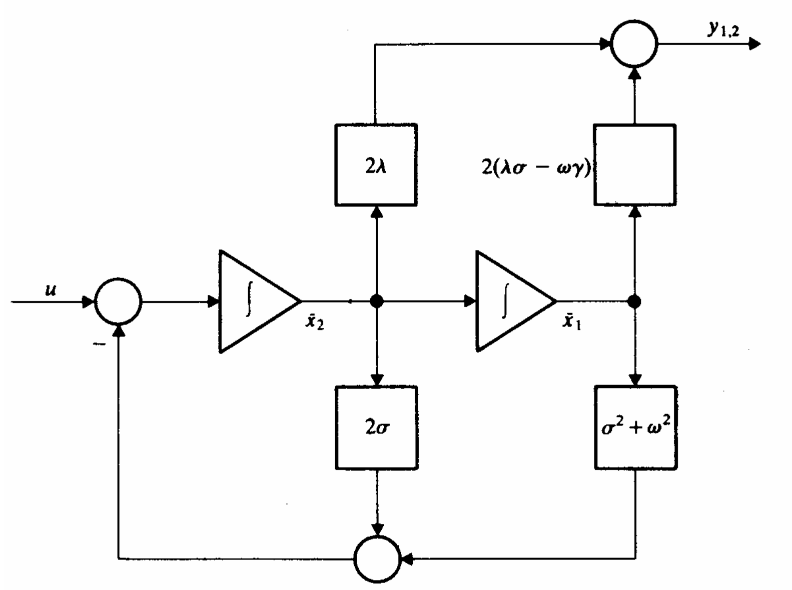


Figure 6.3-2 Companion-Form Realization of Pair of Complex Conjugate Terms as a Real Second-Order Subsystem

When the system has repeated roots, the partial fraction expansion of the transfer function $H(s)$ will be of the form

$$H(s) = b_0 + H_1(s) + \cdots + H_{\bar{k}}(s) \tag{6.3-16}$$

$$H_i(s) = \frac{r_{1i}}{s - s_i} + \frac{r_{2i}}{(s - s_i)^2} + \cdots + \frac{r_{v_i i}}{(s - s_i)^{v_i}} \tag{6.3-17}$$

$$y_i = r_{1i}x_{1i} + r_{2i}x_{2i} + \cdots + r_{v_i i}x_{v_i i} \tag{6.3-18}$$

$$x_{1i} = \frac{u}{s - s_i} \tag{6.3-19}$$

$$x_{2i} = \frac{u}{(s - s_i)^2} = \frac{x_{1i}}{s - s_i} \tag{6.3-20}$$

$$x_{ni} = \frac{x_{n-1i}}{s - s_i} \tag{6.3-21}$$

$$sx_{ni} = s_i x_{ni} + x_{n-1i} \tag{6.3-22}$$

$$\begin{aligned} \dot{x}_{1i} &= s_i x_{1i} + u \\ \dot{x}_{2i} &= x_{1i} + s_i x_{2i} \\ &\ldots\ldots\ldots\ldots\ldots \\ \dot{x}_{v_i i} &= x_{(v_i - 1)i} + s_i x_{v_i i} \end{aligned} \tag{6.3-23}$$

$$A_i = \begin{bmatrix} s_i & 0 & 0 & \cdots & 0 \\ 1 & s_i & 0 & \cdots & 0 \\ 0 & 1 & s_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_i \end{bmatrix}, \; B_i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; C_i = \begin{bmatrix} r_{1i} & r_{2i} & r_{3i} & \cdots & r_{\nu_i i} \end{bmatrix} \tag{6.3-24}$$

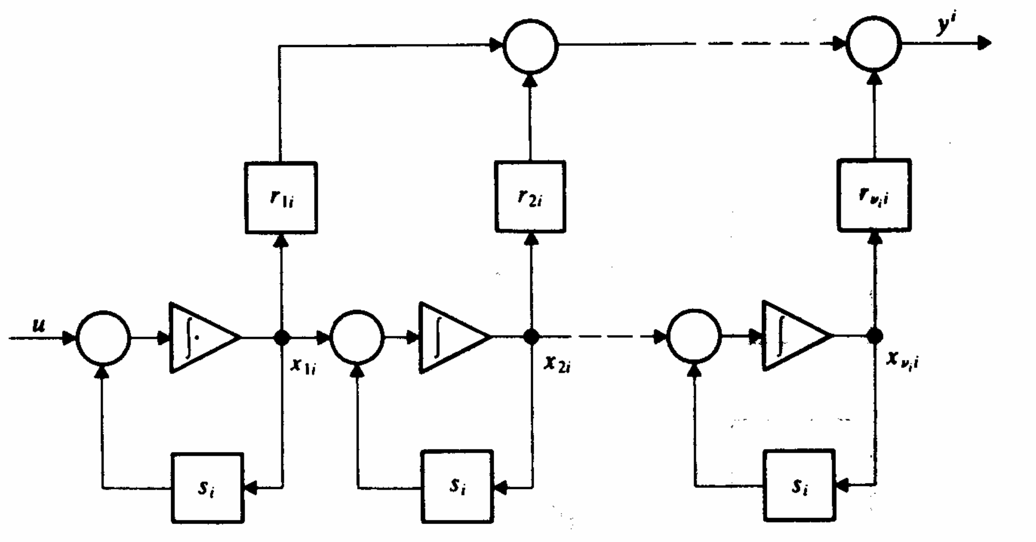


Figure 6.3-3 Jordan-Block Realization of Part of Transfer Function Having Repeated Pole

The state vector of the overall system consists of the concatenation of the state vectors of each of the Jordan blocks.

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^{\bar{k}} \end{bmatrix} \tag{6.3-25}$$

$$A = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{\bar{k}} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_{\bar{k}} \end{bmatrix}, \; C = \begin{bmatrix} C_1 & C_2 & C_3 & \cdots & C_{\bar{k}} \end{bmatrix} \tag{6.3-26}$$

7 Stability

Consider a system whose equilibrium state exists at the origin $x = 0$. The Euclidean length of the vector from the origin, often called the norm, is written as

$$\|x\| = (x'x)^{1/2} = \left(\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right)^{1/2} = (x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2)^{1/2} \qquad (7\text{-}1)$$

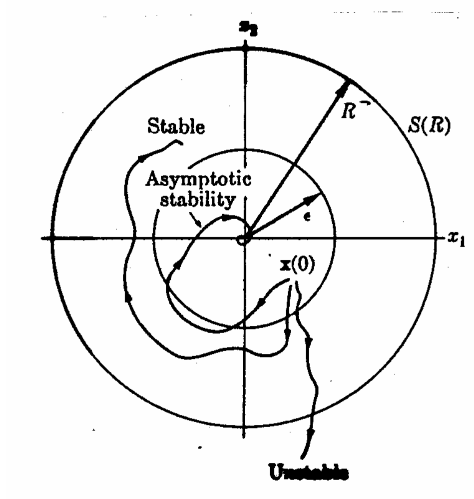


Figure 7-1 Stability Regions in State Space

- Within the n-dimension state space, $S(R)$ is a spherical region of radius R. Then the region $S(R)$ is said to be stable if for any $S(\varepsilon)$ a transient starting in $S(\varepsilon)$ does not leave $S(R)$.
- If there exists a $\delta > 0$, and $x(0)$ is in the sphere $S(\delta)$, and the transient solution approaches the equilibrium state $x = 0$ as time approaches infinity, then the system solution is asymptotically stable.
- If δ can be arbitrarily large then the solution $x = 0$ is asymptotically stable in the large, often called global stability.

7.1 The Direct Method of Liapunov

- The direct method of Liapunov is based on the concept of energy and the relation of stored energy and system stability. The idea is that for a stable system the stored energy will decay with time.
- The energy of a system is a positive quantity and if the time-derivative of the energy is negative we may denote the system as asymptotically stable.

A system is asymptotically stable in some region of the state space if, the Liapunov function V

$$V(x) > 0 \text{ for } x \neq x_e, \quad (7.1\text{-}1)$$

$$dV/dt = \dot{V}(x) < 0 \text{ for } x \neq x_e, \quad (7.1\text{-}2)$$

$$V(x) = 0 \text{ for } x = x_e, \quad (7.1\text{-}3)$$

$$V(x) \to \infty \text{ for } \|x\| \to \infty. \quad (7.1\text{-}4)$$

x_e: the equilibrium state.

Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1}$$

Clearly, the only equilibrium state is the origin, $x = 0$.

Let's choose the following scalar function as a possible Liapunov function:

$$V(x) = 2x_1^2 + x_2^2 \tag{2}$$

which is positive definite function. Then

$$\dot{V}(x) = 4x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1x_2 - 2x_2^2 \tag{3}$$

$\dot{V}(x)$ is indefinite. This implies that this particular function $V(x)$ is not a Liapunov function, and therefore stability cannot be determined by its use.

If we choose the following scalar function as a possible Liapunov function,

$$V(x) = x_1^2 + x_2^2 \tag{4}$$

which is positive definite function. Then

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2x_2^2 \tag{5}$$

which is negative semidefinite. If $\dot{V}(x)$ is to vanish identically for $t \geq t_1$, then x_2 must be zero for all $t \geq t_1$. This requires that $\dot{x}_2 = 0$ for $t \geq t_1$. Since

$$\dot{x}_2 = -x_1 - x_2 \tag{6}$$

x_1 must also be equal to zero for $t \geq t_1$. This means that $\dot{V}(x)$ vanishes identically only at the origin. Hence, the equilibrium state at the origin is asymptotically stable in the large.

If we choose the following scalar function as a possible Liapunov function instead,

$$V(x) = \frac{1}{2}\left[(x_1 + x_2)^2 + 2x_1^2 + x_2^2\right] \tag{7}$$

which is positive definite function. Then

$$\dot{V}(x) = (x_1 + x_2)\dot{x}_1 + 2x_1\dot{x}_1 + (x_1 + x_2)\dot{x}_2 + x_2\dot{x}_2 = -(x_1^2 + x_2^2) \tag{8}$$

which is negative definite. Since $V(x) \to \infty$ as $||x|| \to \infty$, the equilibrium state at the origin is asymptotically stable in the large.

$$\dot{x} = Ax \tag{7.1-5}$$

$$V(x) = x'Px \tag{7.1-6}$$

$$\dot{V}(x) = x'P\dot{x} + \dot{x}'Px \tag{7.1-7}$$

$$\dot{V} = x'PAx + (Ax)'Px = x'(PA + A'P)x \tag{7.1-8}$$

For an asymptotically stable system, $V(x)$ is positive, $\dot{V}(x)$ is negative.

$$\dot{V} = -x'Qx \tag{7.1-9}$$

$$-Q = PA + A'P \tag{7.1-10}$$

For asymptotic stability of a linear system it is sufficient that Q be positive definite.

A necessary condition for a positive definite Q

$$\det[q_{11}],\ \det\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \ldots, \det\begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix} > 0 \quad (7.1\text{-}11)$$

The rate of decay of the Liapunov function, η, the ratio of $-\dot{V}/V$.

$$\eta = \frac{x'Qx}{x'Px} \quad (7.1\text{-}12)$$

$Q = I$,

$$\eta = \frac{\|x\|^2}{x'Px} \quad (7.1\text{-}13)$$

The quotient $\frac{x'Px}{x'x}$, called the Rayleigh quotient, satisfies the relation $\lambda_{\max} \geq \frac{x'Px}{x'x} \geq \lambda_{\min}$, where $\lambda_{\max}$ and $\lambda_{\min}$ are the characteristic roots such that $\lambda_{\max} \geq \lambda_1 \geq \lambda_2 \cdots \geq \lambda_{\min}$.

$$\frac{1}{\lambda_{\max}} \leq \eta \leq \frac{1}{\lambda_{\min}} \quad (7.1\text{-}14)$$

The general solution of P for a linear second-order system

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Q is assigned to be identity matrix I, which is positive definite matrix.

$$PA + A'P = -I$$

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} p_{11}a_{11} + p_{12}a_{21} & p_{11}a_{12} + p_{12}a_{22} \\ p_{12}a_{11} + p_{22}a_{21} & p_{12}a_{12} + p_{22}a_{22} \end{bmatrix} + \begin{bmatrix} a_{11}p_{11} + a_{21}p_{12} & a_{11}p_{12} + a_{21}p_{22} \\ a_{12}p_{11} + a_{22}p_{12} & a_{12}p_{12} + a_{22}p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} p_{11}a_{11} + p_{12}a_{21} + a_{11}p_{11} + a_{21}p_{12} & p_{11}a_{12} + p_{12}a_{22} + a_{11}p_{12} + a_{21}p_{22} \\ p_{12}a_{11} + p_{22}a_{21} + a_{12}p_{11} + a_{22}p_{12} & p_{12}a_{12} + p_{22}a_{22} + a_{12}p_{12} + a_{22}p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2a_{11}p_{11} + 2a_{21}p_{12} & a_{12}p_{11} + (a_{11} + a_{22})p_{12} + a_{21}p_{22} \\ a_{12}p_{11} + (a_{11} + a_{22})p_{12} + a_{21}p_{22} & 2a_{12}p_{12} + 2a_{22}p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

or

$$\begin{bmatrix} 2a_{11} & 2a_{21} & 0 \\ a_{12} & (a_{11} + a_{22}) & a_{21} \\ 0 & 2a_{12} & 2a_{22} \end{bmatrix}\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

so

$$P = \frac{-1}{2(traceA)(\det A)}\begin{bmatrix} (\det A + a_{21}{}^2 + a_{22}{}^2) & -(a_{12}a_{22} + a_{21}a_{11}) \\ -(a_{12}a_{22} + a_{21}a_{11}) & (\det A + a_{11}{}^2 + a_{12}{}^2) \end{bmatrix}$$

where *traceA* is the sum of the diagonal terms of A, $a_{11}+a_{22}$, and det$A$ is the determinnat of A, $a_{11}a_{22}-a_{12}a_{21}$. The system is asymptotically stable if and only if the matrix P is positive definite. Therefore the principal minors of P must be positive,

$$p_{11} = -\frac{\det A + a_{21}{}^2 + a_{22}{}^2}{2(traceA)(\det A)} > 0$$

and

$$\det P = \frac{(a_{11}+a_{22})^2 + (a_{12}-a_{21})^2}{2(traceA)^2(\det A)} > 0$$

Conditions that make the system be stable are

$$\det A = a_{11}a_{22} - a_{12}a_{21} > 0$$

and

$$traceA = a_{11} + a_{22} < 0$$

If we determine the stability of the system by Routh-Hurwitz algorithm, we will find that the conditions that make the system be stable are exactly the same.

Consider again the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1}$$

Clearly, the only equilibrium state is the origin, $x = 0$.

Let's assume a tentative Liapunov function:

$$V(x) = x'Px \tag{2}$$

which P is determined from

$$PA + A'P = -I \tag{3}$$

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4}$$

By expanding this matrix equation, we obtain the following equations.

$$-2p_{12} = -1 \tag{5}$$

$$p_{11} - p_{12} - p_{22} = 0 \tag{6}$$

$$-p_{22} + p_{11} - p_{21} = 0 \tag{7}$$

$$2p_{12} - 2p_{22} = -1 \tag{8}$$

Solving for P, we get

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \tag{9}$$

To test the positive definiteness of P, we check the determinants of the successive principal minors:

$$\frac{3}{2} > 0, \quad \begin{vmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{vmatrix} > 0 \tag{10}$$

Clearly, P is positive definite. Hence the equilibrium state at the origin is asymptotically stable in the large, and a Liapunov function is

$$V(x) = x'Px = \frac{1}{2}\left(3x_1^2 + 2x_1x_2 + 2x_2^2\right) \tag{11}$$

$$\dot{V}(x) = -(x_1^2 + x_2^2) \tag{12}$$

8 Controllability and Observability

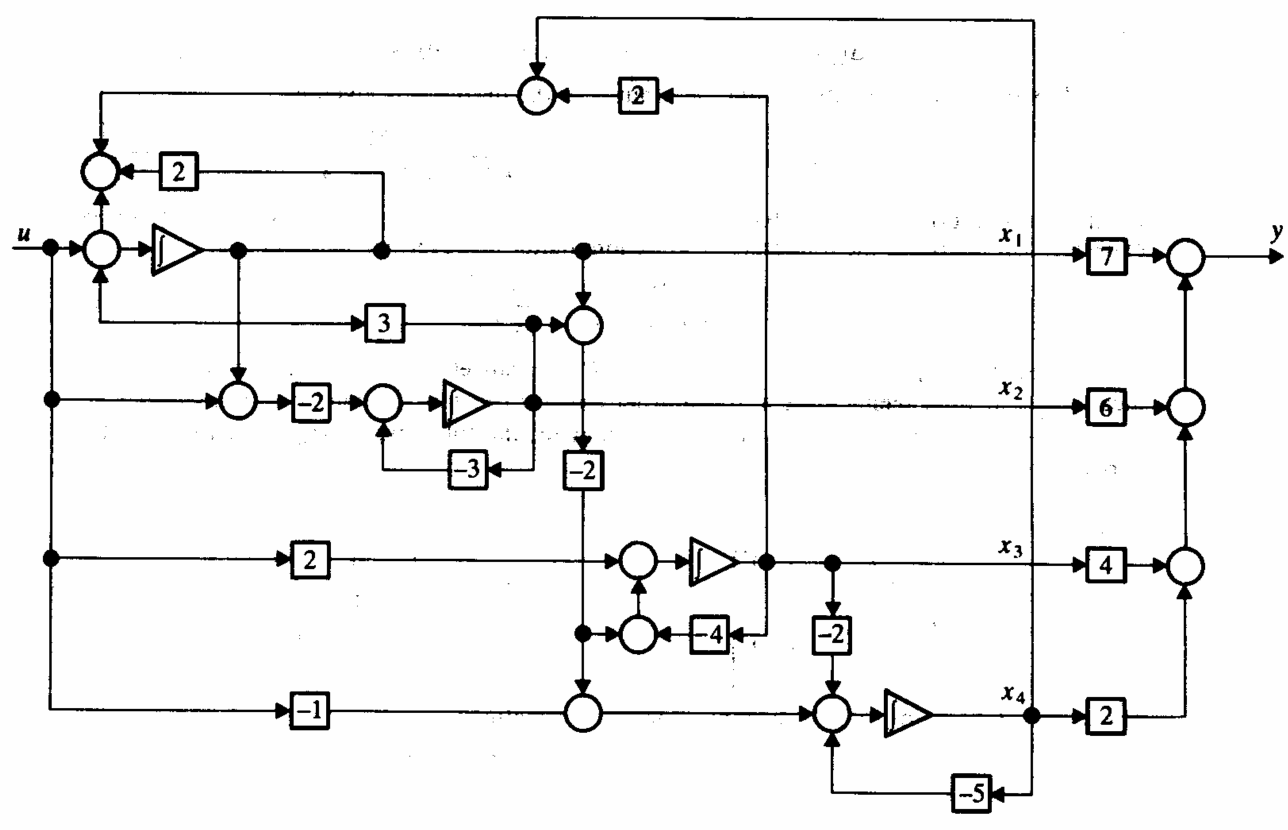


Figure 8-1 Fourth-Order System

Consider the differential equations of a fourth-order system.

$$\begin{aligned}\dot{x}_1 &= 2x_1 + 3x_2 + 2x_3 + x_4 + u\\ \dot{x}_2 &= -2x_1 - 3x_2 - 2u\\ \dot{x}_3 &= -2x_1 - 2x_2 - 4x_3 + 2u\\ \dot{x}_4 &= -2x_1 - 2x_2 - 2x_3 - 5x_4 - u\end{aligned} \qquad (8\text{-}1)$$

$$y = 7x_1 + 6x_2 + 4x_3 + 2x_4 \qquad (8\text{-}2)$$

$$A = \begin{bmatrix} 2 & 3 & 2 & 1\\ -2 & -3 & 0 & 0\\ -2 & -2 & -4 & 0\\ -2 & -2 & -2 & -5\end{bmatrix},\ B = \begin{bmatrix} 1\\ -2\\ 2\\ -1\end{bmatrix},\ C = \begin{bmatrix}7 & 6 & 4 & 2\end{bmatrix} \qquad (8\text{-}3)$$

$$(sI - A)^{-1} = \frac{1}{\Delta(s)}\begin{bmatrix} s^3 + 12s^2 + 47s + 6 & 3s^2 + 21s + 36 & 2s^2 + 14s + 24 & s^2 + 7s + 12\\ -2s^2 - 18s - 40 & s^3 + 7s^2 + 8s - 16 & -4s - 16 & -2s - 18\\ -2s^2 - 12s - 10 & -2s^2 - 12s - 10 & s^3 + 6s^2 + 7s + 2 & -2s - 2\\ -2s^2 - 6s - 4 & -2s^2 - 6s - 4 & -2s^2 - 6s - 4 & s^3 + 5s^2 + 8s + 4\end{bmatrix} \qquad (8\text{-}4)$$

$$\Delta(s) = |sI - A| = s^4 + 21s^3 + 35s^2 + 50s + 24 \qquad (8\text{-}5)$$

$$H(s) = C(sI - A)^{-1}B = \frac{s^3 + 9s^2 + 26s + 24}{s^4 + 21s^3 + 35s^2 + 50s + 24} = \frac{(s+2)(s+3)(s+4)}{(s+1)(s+2)(s+3)(s+4)} = \frac{1}{s+1} \qquad (8\text{-}6)$$

From the transformation of the state variables,

$$\bar{x} = Tx \tag{8-7}$$

$$T = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \tag{8-8}$$

$$\bar{A} = TAT^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad \bar{B} = TB = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{C} = CT^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \tag{8-9}$$

The corresponding equations

$$\begin{aligned} \dot{\bar{x}}_1 &= -\bar{x}_1 + u \\ \dot{\bar{x}}_2 &= -2\bar{x}_2 \\ \dot{\bar{x}}_3 &= -3\bar{x}_3 + u \\ \dot{\bar{x}}_4 &= -4\bar{x}_4 \end{aligned} \tag{8-10}$$

$$y = \bar{x}_1 + \bar{x}_2 \tag{8-11}$$

Only the first subsystem $\bar{x}_1$ contributes to the transfer function $H(s) = 1/(s+1)$.

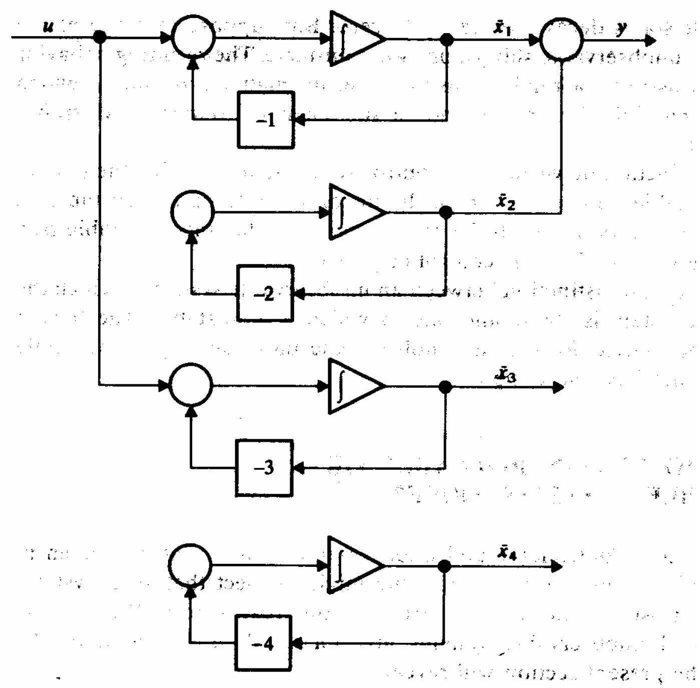


Figure 8-2 System Equivalent of Figure 8-1

$\bar{x}_1$:	affected by the input;	visible in the output
$\bar{x}_2$:	unaffected by the input;	visible in the output
$\bar{x}_3$:	affected by the input;	invisible in the output
$\bar{x}_4$:	unaffected by the input;	invisible in the output

- Transfer function of the system is determined only by the controllable and observable subsystem.
- If the transfer function of a single-input, single-output system is of lower degree than the dimension of the state-space, then the system must contain an uncontrollable subsystem, or an unobservable subsystem, or possibly both.
- If a system contains an uncontrollable subsystem it is said to be uncontrollable.
- If a system contains and unobservable subsystem it is said to be unobservable.
- If at least one of the uncontrollable or unobservable subsystems is unstable, the resulting behavior will be disastrous.

Uncontrollable Causes

1. redundant state variables
2. physically uncontrollable system
3. too much symmetry

Unobservable Cause

1. when state variable is not measured directly and is not fed back to state variables that are measured.

8.1 Definitions and Algebraic Conditions for Controllability and Observability

Definition of Controllability: A system is said to be controllable if and only if its is possible, by means of the input, to transfer the system from any initial state $x(t) = x_t$ to any other state $x_T = x(T)$ in a finite time $T\text{-}t \geq 0$.

Definition of Observability: An unforced system is said to be observable if and only if it is possible to determine any arbitrary initial state $x(t) = x_t$ by using only a finite record, $y(\tau)$ for $t \leq \tau \leq T$, of the output.

Controllability Theorem: A system is controllable if and only if the matrix

$$P(T,t) = \int_t^T \phi(T,\lambda)B(\lambda)B'(\lambda)\phi'(T,\lambda)d\lambda \tag{8.1-1}$$

is nonsingular for some $T > t$, where $\phi(T,t)$ is the state-transition matrix of the system. Matrix $P(T, t)$ is called ***controllability grammian***.

Observability Theorem: A system is observable if and only if the matrix

$$M(T,t) = \int_t^T \Phi'(\lambda,t)C'(\lambda)C(\lambda)\Phi(\lambda,t)d\lambda \tag{8.1-2}$$

is nonsingular for some $T > t$, where $\phi(T,t)$ is the state-transition matrix of the system. Matrix $M(T, t)$ is called ***observability grammian***.

Algebraic Controllability Theorem: The time-invariant system $\dot{x} = Ax + Bu$ is controllable if and only if the rank *r*(*Q*) of the controllability test matrix

$$Q = \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix} \tag{8.1-3}$$

is equal to *k*, the order of the system.

Algebraic Observability Theorem: The unforced time-invariant system $\dot{x} = Ax$ and $y = Cx$ is observable if and only if the rank *r*(*N*) of the observability test matrix

$$N = \begin{bmatrix} C' & A'C' & \cdots & (A')^{k-1}C' \end{bmatrix} \tag{8.1-4}$$

is equal to *k*, the order of the system.

- The algebraic controllability and observability tests are only valid for time-invariant systems.

Rank of the matrix

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ -2 & 4 & -10 & 28 \\ 2 & -6 & 18 & -54 \\ -1 & 3 & -9 & 27 \end{bmatrix} \quad (1)$$

is determined by finding for the maximum non zero determinant matrix which is obtained from crossing row or column of the original matrix.

Since

$$\begin{vmatrix} 1 & -1 & 1 & -1 \\ -2 & 4 & -10 & 28 \\ 2 & -6 & 18 & -54 \\ -1 & 3 & -9 & 27 \end{vmatrix} = 0 \quad (2)$$

rank of this matrix is less than 4.

And since all 16 determinants of the matrices which are obtained from crossing 1 row and 1 column of the original matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \quad (3)$$

rank of this matrix is less than 3.

But since at least

$$\begin{vmatrix} 1 & -1 \\ -2 & 4 \end{vmatrix} \neq 0 \quad (4)$$

rank of this matrix is equal to 2.

Causes of linearly dependency of this matrix:

1. The 4th row is obtained from summation of the 1st and 2nd rows.
2. The 3rd row is obtained from multiplication the 4th row with 2.

Consider controllability of the system in (a)-(c).

(a)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad (1)$$

Determine matrix Q

$$Q = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad (2)$$

Since Q is singular and its rank is 1, this system is uncontrollable.

(b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (3)$$

Determine matrix Q

$$Q = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \tag{4}$$

Since Q is nonsingular and its rank is 2, this system is controllable.

(c)
$$\frac{y(s)}{u(s)} = \frac{s + 2.5}{(s + 2.5)(s - 1)} \tag{5}$$

Clearly, cancellation of the factor (s + 2.5) occurs in the numerator and denominator of this transfer function. Thus, one degree of freedom is lost. Because of this cancellation, this system is either uncontrollable or unobservable.

The same conclusion can be obtained by writing this system in the form of state-space,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \text{ and } y = \begin{bmatrix} 2.5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6}$$

Determine matrix Q

$$Q = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1.5 \end{bmatrix} \tag{7}$$

Since Q is not singular and its rank is 2, this system is controllable.

$$N = \begin{bmatrix} C' & A'C' \end{bmatrix} = \begin{bmatrix} 2.5 & 2.5 \\ 1 & 1 \end{bmatrix} \tag{8}$$

Since N is singular and its rank is 1, this system is unobservable.

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{1}$$

$$y = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{2}$$

Determine matrix Q

$$Q = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix} \tag{3}$$

Since the rank of the matrix Q is 3, this system is controllable

Determine matrix N

$$N = \begin{bmatrix} C' & A'C' & (A')^2C' \end{bmatrix} = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix} \tag{4}$$

Since $|N| = 0$, the rank of the matrix N is less than 3, this system is unobservable

In fact, in this system cancellation occurs in the transfer function of the system. The transfer function between x(s) and $u(s)$ is

$$\frac{x(s)}{u(s)} = \frac{1}{(s+1)(s+2)(s+3)} \tag{5}$$

and the transfer function between $y(s)$ and $x(s)$ is

$$\frac{y(s)}{x(s)} = (s+1)(s+4) \tag{6}$$

Therefore, the transfer function between the output $y(s)$ and the input $u(s)$ is

$$\frac{y(s)}{u(s)} = \frac{(s+1)(s+4)}{(s+1)(s+2)(s+3)} \tag{7}$$

Clearly, the $(s+1)$ cancels each other. This means that there are nonzero initial states $x(0)$, which cannot be determined from the measurement of $y(t)$.

8.2 Disturbances and Tracking Systems: Exogenous Variables

$$\dot{x} = Ax + Bu + Fx_d \tag{8.2-1}$$

x_d : disturbance vector

x_r : reference state vector

$$\dot{x}_d = A_d x_d \tag{8.2-2}$$

$$\dot{x}_r = A_r x_r \tag{8.2-3}$$

$$e = x - x_r \tag{8.2-4}$$

$$\dot{e} = \dot{x} - \dot{x}_r = A(e + x_r) + Fx_d + Bu - Ax_r = Ae + (A - A_r)x_r + Fx_d + Bu = Ae + Ex_0 + Bu \tag{8.2-5}$$

$$E = \left[A - A_r \quad | \quad F \right] \tag{8.2-6}$$

$$x_0 = \begin{bmatrix} x_r \\ \text{---} \\ x_d \end{bmatrix} \tag{8.2-7}$$

x_0 : exogenous input vector

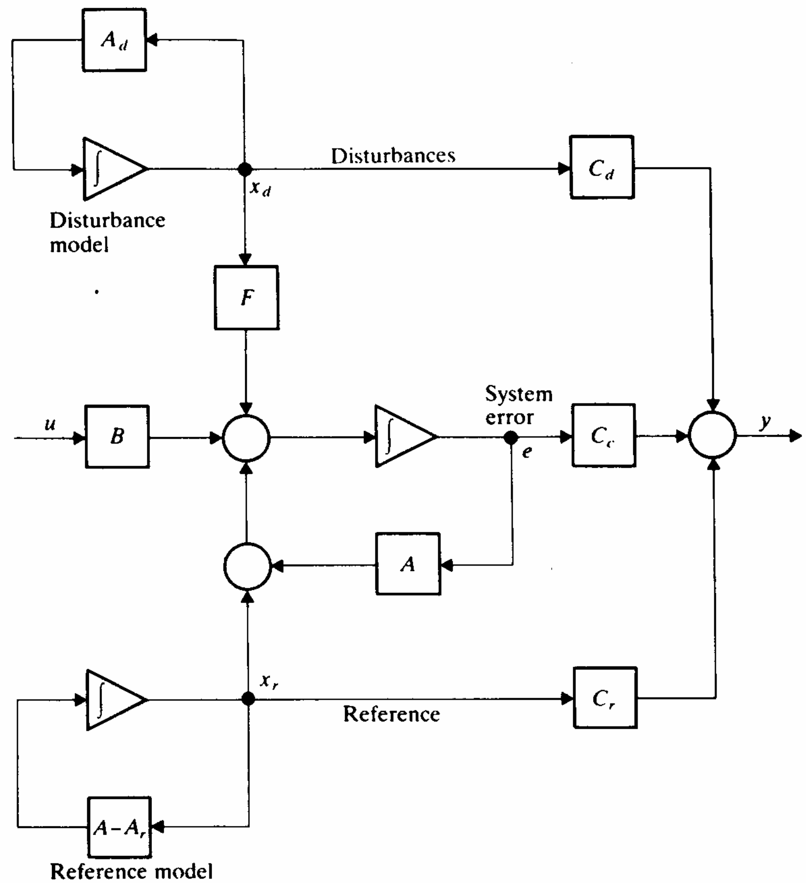


Figure 8.2-1 State-Space Representation of System with Disturbances and Reference Input.

Metastate vector,

$$\mathbf{x} = \begin{bmatrix} e \\ \_\_ \\ x_0 \end{bmatrix} \qquad (8.2\text{-}8)$$

Metastate equation,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad (8.2\text{-}9)$$

$$\mathbf{A} = \begin{bmatrix} A & | & E \\ \_\_ & \_\_ & \_\_ \\ 0 & | & A_0 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} B \\ \_\_ \\ 0 \end{bmatrix},\ \text{and } A_0 = \begin{bmatrix} A_r & | & 0 \\ \_\_ & \_\_ & \_\_ \\ 0 & | & A_d \end{bmatrix} \qquad (8.2\text{-}10)$$

When only the error can be measured, the observation equation

$$y = Ce = \mathbf{C}\mathbf{x} \qquad (8.2\text{-}11)$$

$$\mathbf{C} = \begin{bmatrix} C & | & 0 & 0 \end{bmatrix} \qquad (8.2\text{-}12)$$

When it is possible to measure the error, the reference state, and the disturbance state, the observation equation

$$y = C_e e + C_r x_r + C_d x_d \qquad (8.2\text{-}13)$$

$$\mathbf{C} = \begin{bmatrix} C_e & | & C_r & C_d \end{bmatrix} \qquad (8.2\text{-}14)$$

- The subsystems for the disturbance x_d and the reference x_r are clearly not controllable.
- With C_d and C_r present, the system is likely to be observable. But even if only C_e is present, the system may be observable because there is a path from x_r to the output through the subsystem that generates the error.

9 Shaping the Dynamic Response

- In ***pole-placement*** method, it is possible to place the closed-loop poles anywhere in the complex s plane.
- All the state variables must be accessible for measurement or estimated from measured output.

$$u = -Gx \tag{9-1}$$

G: gain matrix in a linear feedback law

$$u = -G\hat{x} \tag{9-2}$$

$\hat{x}$: state vector of the ***observer***, estimation of the state vector

9.1 Design of Regulators for Single-Input, Single-Output Systems

$$G = g' = \begin{bmatrix} g_1 & g_2 & \cdots & g_k \end{bmatrix} \tag{9.1-1}$$

$$\dot{x} = Ax + Bu \tag{9.1-2}$$

$$B = b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \tag{9.1-3}$$

With the control law $u = -Gx = -g'x$,

$$\dot{x} = (A - bg')x \tag{9.1-4}$$

- Our objective is to find the matrix $G = g'$ which places the closed-loop dynamics matrix at the locations desired.

$$A_c = A - bg' \tag{9.1-5}$$

- There are k gains and k poles for a kth order system, so there are precisely as many gains as needed to specify each of the closed-loop poles.

$$|sI - A_c| = |sI - A + bg'| = s^k + \hat{a}_1 s^{k-1} + \cdots + \hat{a}_k \tag{9.1-6}$$

In the first companion form and left to right numbering of state variables,

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{9.1-7}$$

$$bg' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} g_1 & g_2 & g_3 & \cdots & g_k \end{bmatrix} = \begin{bmatrix} g_1 & g_2 & g_3 & \cdots & g_k \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{9.1-8}$$

$$A_c = A - bg' = \begin{bmatrix} -a_1 - g_1 & -a_2 - g_2 & \cdots & -a_{k-1} - g_{k-1} & -a_k - g_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} -\hat{a}_1 & -\hat{a}_2 & \cdots & -\hat{a}_{k-1} & -\hat{a}_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad (9.1\text{-}9)$$

$$a_i + g_i = \hat{a}_i \text{ or } g = \hat{a} - a \quad (9.1\text{-}10)$$

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \text{and } \hat{a} = \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_k \end{bmatrix} \quad (9.1\text{-}11)$$

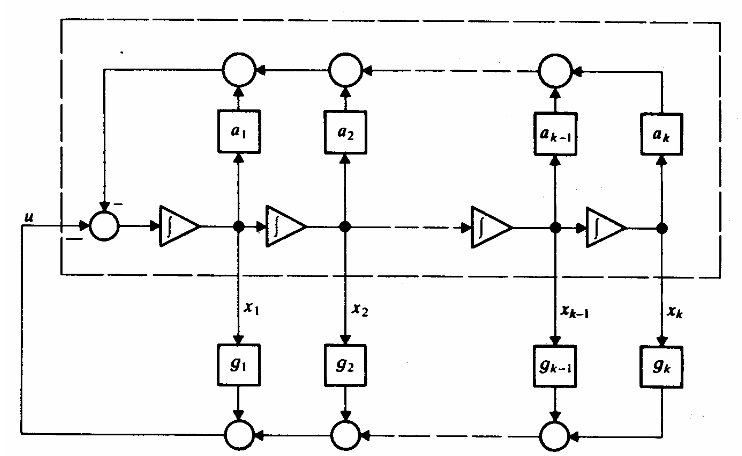


Figure 9.1-1 State Variable Feedback for a System in First Companion Form

- Transformation of state vector from original form, x, to the first companion form, $\bar{x}$,

$$\bar{x} = Tx \tag{9.1-12}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \tag{9.1-13}$$

$$\bar{A} = TAT^{-1} \text{ and } \bar{b} = Tb \tag{9.1-14}$$

$$\bar{g} = \hat{a} - \bar{a} = \hat{a} - a \tag{9.1-15}$$

$$u = -g'x = -g'T^{-1}\bar{x} = -\bar{g}'\bar{x} \tag{9.1-16}$$

$$\bar{g}' = g'T^{-1} \tag{9.1-17}$$

$$g = T'\bar{g} = T'(\hat{a} - a) \tag{9.1-18}$$

- The desired transformation matrix T is the product of two matrices V and U:

$$T = VU \tag{9.1-19}$$

- The first matrix transforms the original system into an intermediate system in the second companion form.
- The second transformation transforms the intermediate system into the first companion form.

$$\tilde{x} = Ux \tag{9.1-20}$$

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{b}u \tag{9.1-21}$$

$$\tilde{A} = UAU^{-1} \text{ and } \tilde{b} = Ub \tag{9.1-22}$$

- U is the inverse of controllability test matrix Q.

$$U = Q^{-1} = \begin{bmatrix} b & Ab & \cdots & A^{k-1}b \end{bmatrix}^{-1} \tag{9.1-23}$$

$$\bar{x} = V\tilde{x} \tag{9.1-24}$$

$$\bar{A} = V\tilde{A}V^{-1} \text{ and } \bar{b} = V\tilde{b} \tag{9.1-25}$$

- V^{-1} is the transpose of the upper left-hand k-by-k submatrix of the triangular Toeplitz matrix .

$$V = W^{-1} = \begin{bmatrix} 1 & a_1 & \cdots & a_{k-1} \\ 0 & 1 & \cdots & a_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}^{-1} \tag{9.1-26}$$

$$g = (VU)'(\hat{a} - a) \tag{9.1-27}$$

$$V = W^{-1} \text{ and } U = Q^{-1} \tag{9.1-28}$$

$$VU = W^{-1}Q^{-1} = (QW)^{-1} \tag{9.1-29}$$

$$g = [(QW)']^{-1}(\hat{a} - a) \tag{9.1-30}$$

A dc motor driving an inertial load constitutes a simple instrument servo for keeping the load at a fixed position. The state-space equations for the motor-driven inertia

$$\dot{\theta} = \omega \tag{9.1-31}$$

$$\dot{\omega} = -\alpha\omega + \beta u \tag{9.1-32}$$

θ : angular position

ω : angular velocity

u : the applied voltage,

$$\alpha = -\frac{K_1 K_2}{JR} \text{ and } \beta = \frac{K_1}{JR} \tag{9.1-33}$$

If the desired position θ_r is constant.

$$x_r = \begin{bmatrix} \theta_r \\ 0 \end{bmatrix} \tag{9.1-34}$$

$$\dot{x}_r = A_r x_r = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x_r \tag{9.1-35}$$

$$e = x - x_r = \begin{bmatrix} \theta - \theta_r \\ \omega \end{bmatrix} \tag{9.1-36}$$

$$\dot{e} = Ae + Bu + [A - Ar]x_r \tag{9.1-37}$$

$$\begin{bmatrix} \dot{e} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} e \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u + \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} \theta_r \\ 0 \end{bmatrix} \tag{9.1-38}$$

$$\begin{bmatrix} \dot{e} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} e \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u \tag{9.1-39}$$

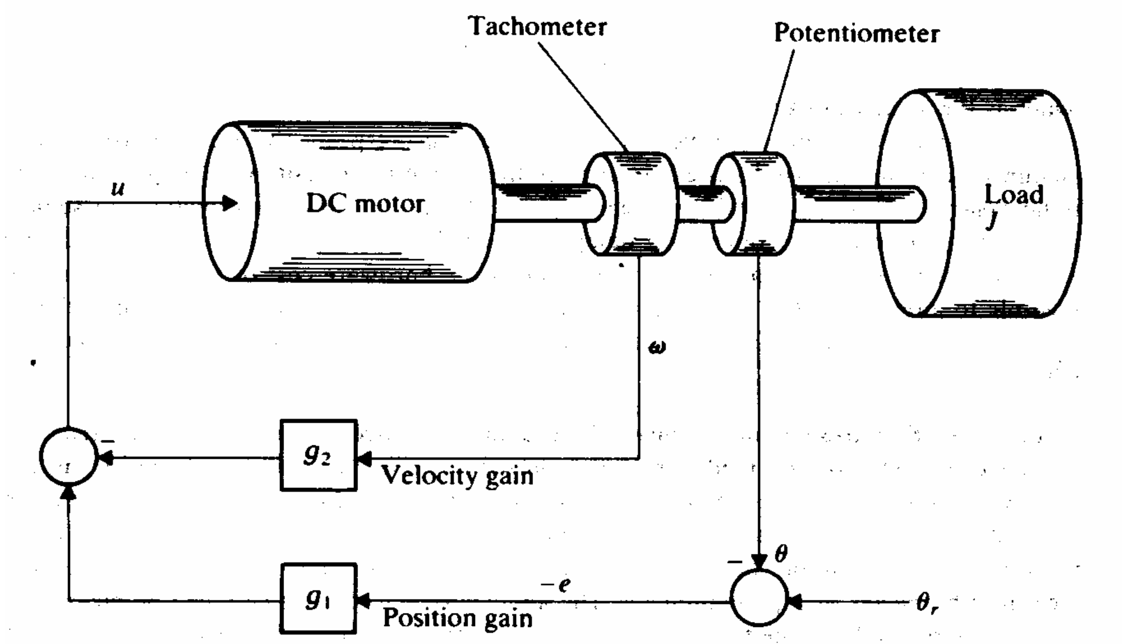


Figure 9.1-2 Implementation of an Instrument Servo

The characteristic polynomial of the system

$$|sI - A| = \begin{vmatrix} s & -1 \\ 0 & s+\alpha \end{vmatrix} = s^2 + \alpha s \tag{9.1-40}$$

The desired characteristic polynomial of the system

$$|sI - A_c| = s^2 + \hat{a}_1 s + \hat{a}_2 \tag{9.1-41}$$

$$a = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \tag{9.1-42}$$

$$\hat{a} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} \tag{9.1-43}$$

$$Q = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \text{and } W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \tag{9.1-44}$$

$$QW = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} = (QW)' \tag{9.1-45}$$

$$[(QW)']^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \tag{9.1-46}$$

$$g = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \begin{bmatrix} \hat{a}_1 - \alpha \\ \hat{a}_2 \end{bmatrix} = \begin{bmatrix} \hat{a}_2/\beta \\ (\hat{a}_1 - \alpha)/\beta \end{bmatrix} \tag{9.1-47}$$

For a control law of the form

$$u = -g_1 e - g_2 \omega \quad (9.1\text{-}48)$$

$$\dot{e} = \omega \quad (9.1\text{-}49)$$

$$\dot{\omega} = -g_1 \beta e - (\alpha + \beta g_2)\omega \quad (9.1\text{-}50)$$

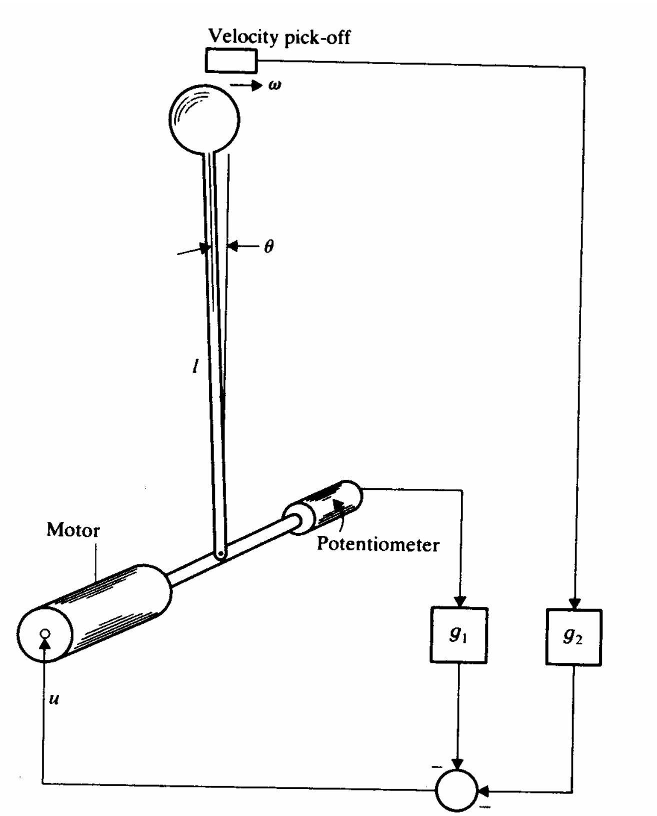
$$A_c = \begin{bmatrix} 0 & 1 \\ -g_1\beta & -(\alpha + g_2\beta) \end{bmatrix} \quad (9.1\text{-}51)$$

$$|sI - A_c| = s^2 + (\alpha + g_2\beta)s + g_1\beta \quad (9.1\text{-}52)$$

$$\hat{a}_1 = \alpha + g_2\beta \text{ and } \hat{a}_2 = g_1\beta \quad (9.1\text{-}53)$$

$$g_1 = \hat{a}_2 / \beta \text{ and } g_2 = (\hat{a}_1 - \alpha)/\beta \quad (9.1\text{-}54)$$

An inverted pendulum can be stabilized by a closed-loop feedback system. A possible control system implementation is shown in figure below for a pendulum constrained to rotate about a shaft at its bottom point.



The dynamic equations governing the inverted pendulum in which the point of attachment does not translate is given by

$$[J_m + ml^2]\dot{\omega} = mgl\sin\theta - \frac{K_1K_2}{R}\omega + \frac{K_1}{R}u \approx mgl\theta - \frac{K_1K_2}{R}\omega + \frac{K_1}{R}u \tag{1}$$

$$\dot{\theta} = \omega \tag{2}$$

$$\dot{\omega} = \Omega^2\theta - \alpha\omega + \beta u \tag{3}$$

where

$$\alpha = -\frac{K_1K_2}{JR} \text{ and } \beta = \frac{K_1}{JR} \tag{4}$$

with the inertial J being the total reflected inertia:

$$J = J_m + ml^2 \tag{5}$$

and m is the pendulum bob mass, l is the distance of the bob from the pivot.

The natural frequency Ω is given by

$$\Omega^2 = \frac{mgl}{J_m + ml^2} = \frac{g}{l + J_m / ml} \tag{6}$$

Since the linearization is valid only when the pendulum is nearly vertical, we shall assume that the control objective is to maintain $\theta = 0$. Thus we have a simple regulator problem.

The matrices A and B are

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \beta \end{bmatrix} \tag{7}$$

The open-loop characteristic polynomial is

$$|sI - A| = \begin{vmatrix} s & -1 \\ -\Omega^2 & s+\alpha \end{vmatrix} = s^2 + \alpha s - \Omega^2 \tag{8}$$

Thus

$$a_1 = \alpha \text{ and } a_2 = -\Omega^2 \tag{9}$$

The open-loop system is unstable.

The controllability test matrix and the W matrix are

$$Q = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \text{ and } W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \tag{10}$$

And

$$[(QW)']^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \tag{11}$$

Thus the gain matrix required for pole placement is

$$g = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \begin{bmatrix} \hat{a}_1 - \alpha \\ \hat{a}_2 + \Omega^2 \end{bmatrix} = \begin{bmatrix} (\hat{a}_2 + \Omega^2)/\beta \\ (\hat{a}_1 - \alpha)/\beta \end{bmatrix} \tag{12}$$

$$A_c = \begin{bmatrix} 0 & 1 \\ \Omega^2 - g_1\beta & -(\alpha + g_2\beta) \end{bmatrix} \tag{13}$$

$$\left| sI - A_c \right| = s^2 + (\alpha + g_2\beta)s + g_1\beta - \Omega^2 \tag{14}$$

$$\hat{a}_1 = \alpha + g_2\beta \text{ and } \hat{a}_2 = g_1\beta - \Omega^2 \tag{15}$$

$$g_1 = (\hat{a}_2 + \Omega^2)/\beta \text{ and } g_2 = (\hat{a}_1 - \alpha)/\beta \tag{16}$$

9.2 Multiple-Input System

- In a controllable system with multiple inputs, there will be more gains available than are needed to place all of the closed-loop poles.
- It is possible to specify all the closed-loop poles and still be able to satisfy other requirements.
- Design approaches of multiple-input system
    - Setting some of the gains to zero.
    - Selecting a particular structure for the gain matrix to make each control variable depend on a different group of state variables which are physically more closely related to that control variable than to the other control variables

Consider the system defined by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{1}$$

If we want to find a state-feedback gain matrix such that the closed-loop system has eigenvalues at -1, -2, and -3,

$$|sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & -2 & s-3 \end{vmatrix} = s^3 - 3s^2 - 2s - 1 = s^3 + a_1 s^2 + a_2 s + a_3 \tag{2}$$

Or

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} \tag{3}$$

Currently the system has eigenvalues at 3.63, -0.31±0.42i and unstable.

Consider the effect from the first input u_1,

$$Q_1 = [b_1 \quad Ab_1 \quad A^2 b_1] = \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] \tag{4}$$

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad (5)$$

Since Q_1 is not singular matrix, we can find state-feedback gains from the input u_1.

$$W = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \quad (6)$$

$$Q_1 W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (7)$$

$$\left[(Q_1 W)^{-1}\right]' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \quad (8)$$

Characteristic equation of the closed-loop system is obtained from

$$| sI - A_c | = (s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6 = s^3 + \hat{a}_1 s^2 + \hat{a}_2 s + \hat{a}_3 \quad (9)$$

Or

$$\hat{a} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 6 \end{bmatrix} \quad (10)$$

Thus,

$$g_1 = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \left[(Q_1 W)^{-1}\right]' \begin{bmatrix} \hat{a}_1 - a_1 \\ \hat{a}_2 - a_2 \\ \hat{a}_3 - a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6+3 \\ 11+2 \\ 6+1 \end{bmatrix} = \begin{bmatrix} 9 \\ 25 \\ 40 \end{bmatrix} \quad (11)$$

Since by only the input u_1, the compensated system meets the requirement already, the gains of the state feedback from the other input u_2 are zeros.

Thus,

$$G = \begin{bmatrix} 9 & 25 & 40 \\ 0 & 0 & 0 \end{bmatrix} \quad (12)$$

Alternative solution

Consider the effect from the second input u_2,

$$Q_2 = [b_2 \quad Ab_2 \quad A^2 b_2] = \begin{bmatrix} 0 & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ 1 & & \\ 0 & & \end{bmatrix} \quad (13)$$

$$Q_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 7 \end{bmatrix} \quad (14)$$

Since Q_2 is not singular matrix, we can find state-feedback gains from the input u_2.

$$W = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

$$Q_2 W = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 7 \end{bmatrix} \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 0 \\ 0 & 2 & 1 \end{bmatrix} \tag{16}$$

$$\left[(Q_2 W)^{-1}\right]' = \begin{bmatrix} 0.43 & 1 & 1.29 \\ 0.14 & 0 & 0.43 \\ -0.29 & 0 & 0.14 \end{bmatrix}' = \begin{bmatrix} 0.43 & 0.14 & -0.29 \\ 1 & 0 & 0 \\ 1.29 & 0.43 & 0.14 \end{bmatrix} \tag{17}$$

Characteristic equation of the closed-loop system is obtained from

$$| sI - A_c | = (s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6 = s^3 + \hat{a}_1 s^2 + \hat{a}_2 s + \hat{a}_3 \tag{18}$$

Or

$$\hat{a} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 6 \end{bmatrix} \tag{19}$$

Thus,

$$g_2 = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \left[(Q_2 W)^{-1}\right]' \begin{bmatrix} \hat{a}_1 - a_1 \\ \hat{a}_2 - a_2 \\ \hat{a}_3 - a_3 \end{bmatrix} = \begin{bmatrix} 0.43 & 0.14 & -0.29 \\ 1 & 0 & 0 \\ 1.29 & 0.43 & 0.14 \end{bmatrix} \begin{bmatrix} 6+3 \\ 11+2 \\ 6+1 \end{bmatrix} = \begin{bmatrix} 3.71 \\ 9 \\ 18.14 \end{bmatrix} \tag{20}$$

Since by only the input u_2, the compensated system meets the requirement already, the gains of the state feedback from the other input u_1 are zeros.

Thus,

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 3.71 & 9 & 18.14 \end{bmatrix} \tag{21}$$

Alternative solution

If the requirement that u_1 depends on x_1 and x_2 whereas u_2 depends on x_3 only, thus

$$G = \begin{bmatrix} g_{11} & g_{12} & 0 \\ 0 & 0 & g_{23} \end{bmatrix} \tag{22}$$

Since

$$A_c = A - BG = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & 0 \\ 0 & 0 & g_{23} \end{bmatrix} = \begin{bmatrix} -g_{11} & 1-g_{12} & 0 \\ 0 & 0 & 1-g_{23} \\ 1 & 2 & 3 \end{bmatrix} \tag{23}$$

$$|sI - A_c| = \begin{vmatrix} s+g_{11} & g_{12}-1 & 0 \\ 0 & s & g_{23}-1 \\ -1 & -2 & s-3 \end{vmatrix} = s^3 + (g_{11}-3)s^2 + (2g_{23} - 3g_{11} - 2)s + 2g_{23}g_{11} - g_{12}g_{13} - 2g_{11} + g_{12} + g_{23} - 1 \tag{24}$$

Characteristic equation of the closed-loop system is obtained from

$$|sI - A_c| = (s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6 = s^3 + \hat{a}_1 s^2 + \hat{a}_2 s + \hat{a}_3 \tag{25}$$

Or

$$g_{11} - 3 = 6; \quad g_{11} = 9 \tag{26}$$

$$2g_{23} - 3g_{11} - 2 = 2g_{23} - 3(9) - 2 = 11; \quad g_{23} = 20 \tag{27}$$

$$2g_{23}g_{11} - g_{12}g_{23} - 2g_{11} + g_{12} + g_{23} - 1 = 2(20)(9) - 20g_{12} - 2(9) + g_{12} + 20 - 1 = 6; \quad g_{12} = 18.68 \tag{28}$$

Thus,

$$G = \begin{bmatrix} 9 & 18.68 & 0 \\ 0 & 0 & 20 \end{bmatrix} \tag{29}$$

9.3 Disturbances and Tracking Systems: Exogenous Variables

$$e = x - x_r \tag{9.3-1}$$

$$\dot{x}_r = A_r x_r \tag{9.3-2}$$

$$\dot{x}_d = A_d x_d \tag{9.3-3}$$

$$\dot{e} = Ae + (A - A_r)x_r + Fx_d + Bu = Ae + Bu + Ex_0 \tag{9.3-4}$$

A linear control law,

$$u = -Ge - G_0 x_0 = -Ge - G_r x_r - G_d x_d \tag{9.3-5}$$

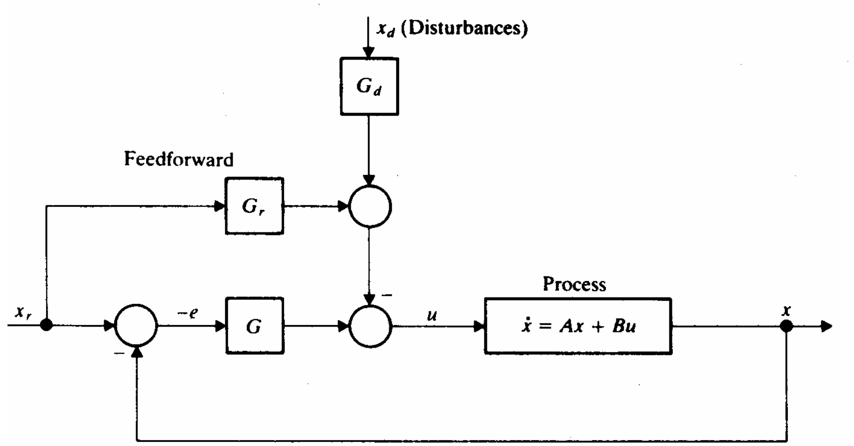


Figure 9.3-1 Schematic of Feedback System for Process with Reference State and Disturbance Input.

- The design is based on the assumption that the exogenous input vector x_0 as well as the system error e are accessible for measurement during the operation of the control system.

The closed-loop dynamics

$$\dot{e} = Ae + Ex_0 - B(Ge + G_0 x_0) \tag{9.3-6}$$

- The most desirable is to choose the gains G and G_0 to keep the system error zero, which is not possible in the system that has number of control inputs less than number of state variables.
- More reasonable performance objectives, having number of control inputs not less than measured output, are the followings:

  (a) The closed-loop system should be asymptotically stable.

  (b) The measured output is zero at the steady state.

The steady-state condition is characterized by a constant error state vector,

$$\dot{e} = Ae + Ex_0 - B(Ge + G_0 x_0) = 0 \tag{9.3-7}$$

$$(A - BG)e = (BG_0 - E)x_0 \tag{9.3-8}$$

$$e = (A - BG)^{-1}(BG_0 - E)x_0 \tag{9.3-9}$$

- When the number of control inputs equals to number of state variables and $(A - BG)^{-1}B$ is a square matrix and invertible, e can be controlled to zero at steady state.

$$0 = (A - BG)^{-1}(BG_0 - E)x_0 \tag{9.3-10}$$

$$(A - BG)^{-1} E = (A - BG)^{-1} BG_0 \tag{9.3-11}$$

$$G_0 = [(A - BG)^{-1} B]^{-1} (A - BG)^{-1} E \tag{9.3-12}$$

- When the number of control inputs equals to number of measured outputs and $C(A - BG)^{-1} B$ is a square matrix and invertible, y can be controlled to zero at steady state.

$$y = Ce = C(A - BG)^{-1}(BG_0 - E)x_0 = 0 \tag{9.3-13}$$

$$C(A - BG)^{-1} BG_0 = C(A - BG)^{-1} E \tag{9.3-14}$$

$$G_0 = [C(A - BG)^{-1} B]^{-1} C(A - BG)^{-1} E \tag{9.3-15}$$

$$G_0 = B^{\#} E \tag{9.3-16}$$

when $B^{\#} = [C(A - BG)^{-1} B]^{-1} C(A - BG)^{-1}$.

Example: Consider a state-space a system represented by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} x_d \tag{9.3-17}$$

$$y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{9.3-18}$$

Assume that the desired (reference) state is

$$x_{1r} = \bar{x}_1 = \text{const} \tag{9.3-19}$$

$$x_{2r} = \bar{x}_2 = \text{const} \tag{9.3-20}$$

$$A_r = 0 \tag{9.3-21}$$

When the characteristic equation of $s^2 + \hat{a}_1 s + \hat{a}_2 = 0$ is desired, by pole placement method,

$$G = \begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} \dfrac{a_{21}b_1\hat{a}_1 + a_{11}a_{21}b_1 + a_{21}a_{22}b_1 - a_{22}b_2\hat{a}_1 - a_{22}^2 b_2 - b_2\hat{a}_2 - a_{11}a_{22}b_2}{a_{21}b_1^2 - a_{12}b_2^3} & \dfrac{a_{22}b_1\hat{a}_1 - a_{11}a_{12}b_2 - a_{12}a_{22}b_2 - a_{12}b_2\hat{a}_1 + a_{22}^2 b_1 + b_1\hat{a}_2 + a_{11}a_{22}b_1}{a_{21}b_1^2 - a_{12}b_2^3} \end{bmatrix} \tag{9.3-22}$$

$$G_0 = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1}E = B^{\#}E \tag{9.3-23}$$

$$A_c^{-1} = (A - BG)^{-1} = \begin{bmatrix} a_{11} - b_1 g_1 & a_{12} - b_1 g_2 \\ a_{21} - b_2 g_1 & a_{22} - b_2 g_2 \end{bmatrix}^{-1} \tag{9.3-24}$$

$$A_c^{-1} = \frac{1}{(a_{11} - b_1 g_1)(a_{22} - b_2 g_2) - (a_{12} - b_1 g_2)(a_{21} - b_2 g_1)} \begin{bmatrix} a_{22} - b_2 g_2 & -(a_{12} - b_1 g_2) \\ -(a_{21} - b_2 g_1) & a_{11} - b_1 g_1 \end{bmatrix} \tag{9.3-25}$$

$$C(A - BG)^{-1}B = \frac{(c_1(a_{22} - b_2 g_2) - c_2(a_{21} - b_2 g_1))b_1 - (c_1(a_{12} - b_1 g_2) - c_2(a_{11} - b_1 g_1))b_2}{(a_{11} - b_1 g_1)(a_{22} - b_2 g_2) - (a_{12} - b_1 g_2)(a_{21} - b_2 g_1)} \tag{9.3-26}$$

$$C(A - BG)^{-1}B = \frac{(c_1 a_{22} - c_2 a_{21})b_1 - (c_1 a_{12} - c_2 a_{11})b_2}{(a_{11} - b_1 g_1)(a_{22} - b_2 g_2) - (a_{12} - b_1 g_2)(a_{21} - b_2 g_1)} = p \tag{9.3-27}$$

$$B^{\#} = \frac{1}{p} \cdot \frac{1}{(a_{11} - b_1 g_1)(a_{22} - b_2 g_2) - (a_{12} - b_1 g_2)(a_{21} - b_2 g_1)} \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} a_{22} - b_2 g_2 & -(a_{12} - b_1 g_2) \\ -(a_{21} - b_2 g_1) & a_{11} - b_1 g_1 \end{bmatrix} \tag{9.3-28}$$

$$B^{\#} = \frac{1}{(c_1 a_{22} - c_2 a_{21})b_1 - (c_1 a_{12} - c_2 a_{11})b_2} \begin{bmatrix} c_1(a_{22} - b_2 g_2) - c_2(a_{21} - b_2 g_1) & c_2(a_{11} - b_1 g_1) - c_1(a_{12} - b_1 g_2) \end{bmatrix} \tag{9.3-29}$$

$$G_d = B^{\#}F = \frac{(c_1(a_{22} - b_2 g_2) - c_2(a_{21} - b_2 g_1))f_1 + (c_2(a_{11} - b_1 g_1) - c_1(a_{12} - b_1 g_2))f_2}{(c_1 a_{22} - c_2 a_{21})b_1 - (c_1 a_{12} - c_2 a_{11})b_2} \tag{9.3-30}$$

9.4 Where Should the Closed-Loop Poles Be Placed?

$$g = [(QW)']^{-1}(\hat{a} - a) \tag{9.4-1}$$

- The larger the gain, the larger the control input.
- The actuator which supplies the control u cannot be arbitrarily large without incurring penalties of cost and weight.
- Limiting the control may be to avoid the potential damaging effects of stresses on the process that large inputs might cause.
- If the control signal is larger than possible or permissible for reasons of safety, the actuator will ***saturate*** at a lower input level.
- The less the poles are moved, the smaller the gain matrix.
- The less controllable the system, the larger the gains that are needed to effect a change in the system poles.
- Efficient use of the control signal would require that all the closed-loop poles be about the same distance from the origin.

The choice of closed-loop poles:

- Select a bandwidth high enough to achieve the desired speed of response.
- Keep the bandwidth low enough to avoid exciting unmodeled high-frequency effects and undesired response to noise.
- Place the poles at approximately uniform distances from the origin for efficient use of the control effort.

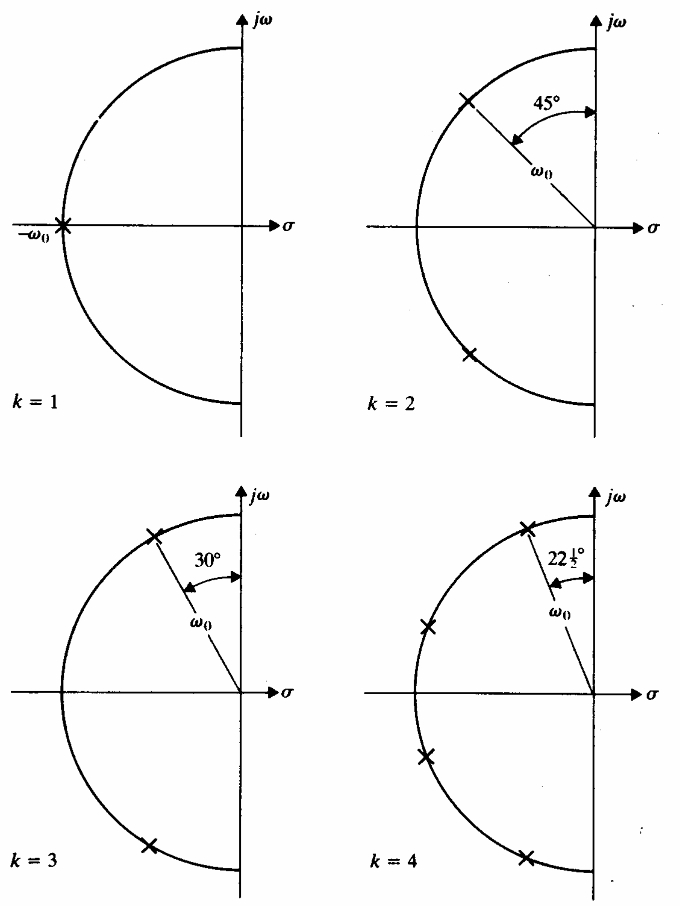


Figure 9.4-1 Butterworth Pole Configuration

- The Butterworth polynomials $B_k(z), z = s/\omega_0$ of the first few are

$$B_1(z) = z + 1 \tag{9.4-2}$$

$$B_2(z) = z^2 + \sqrt{2}z + 1 \tag{9.4-3}$$

$$B_3(z) = z^3 + 2z^2 + 2z + 1 \tag{9.4-4}$$

$$B_4(z) = z^4 + 2.613z^3 + (2 + \sqrt{2})z^2 + 2.6132z + 1 \tag{9.4-5}$$

- As the number of poles becomes high, one pair of poles comes precariously close to the imaginary axis. It might be desirable to move these poles farther into the left half-plane.

10 Linear Observer

- In order to place the poles at arbitrary locations, it is necessary to have all the state variables available for feedback.
- If the system is observable, it is possible to estimate the state variables that are not directly accessible to measurement using the measurement data.
- State-variable estimates may be even preferable to direct measurements, because the errors introduced by the instruments may be larger than the errors in estimating these variables.
- A dynamic system whose state variables are the estimates of the state variables is called an **observer**.
- For any observable linear system, an observer can be designed having the property that the estimation error can be made to go to zero as fast as one may desire. The design technique is equivalent to pole placement in feedback system design.

10.1 Structure and Properties of Observers

A dynamic system in state-space representation

$$\dot{x} = Ax + Bu \tag{10.1-1}$$

- A control law $u = -Gx$ can be used only x is acccssible for measurement.
- Instead of being able to measure the state x, only the output is measurable.

$$y = Cx + Du \tag{10.1-2}$$

An estimation of $x(t)$, $\hat{x}(t)$, follows a dynamic of the observer.

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u + Ky \tag{10.1-3}$$

$$e = x - \hat{x} \tag{10.1-4}$$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - \hat{A}(x - e) - \hat{B}u - K(Cx + Du) = \hat{A}e + (A - KC - \hat{A})x + (B - KD - \hat{B})u \tag{10.1-5}$$

$$\hat{A} = A - KC \tag{10.1-6}$$

$$\hat{B} = B - KD \tag{10.1-7}$$

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky = A\hat{x} + Bu + K(y - C\hat{x} - Du) \tag{10.1-8}$$

- The difference between the actual measurement y and the estimated measurement is often called the ***residual***.

$$r = y - C\hat{x} - Du = C(x - \hat{x}) = Ce \tag{10.1-9}$$

- In most of the systems, the measured output depends only on state variables not input, thus, $y = Cx$.

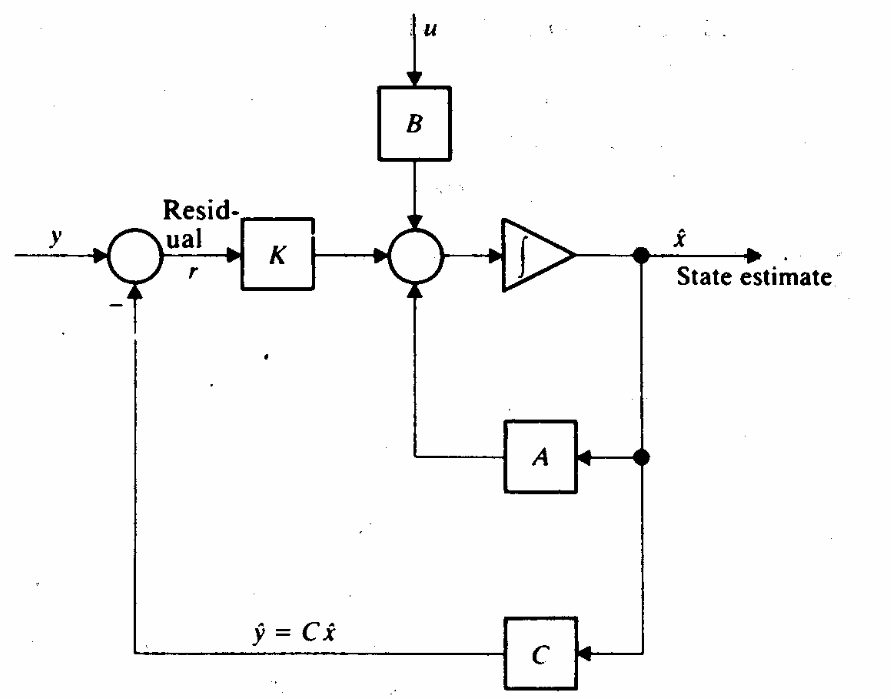


Figure 10.1-1 Block Diagram of Linear Observer

$$\dot{e} = \hat{A}e \qquad (10.1\text{-}10)$$

$$e(s) = (sI - \hat{A})^{-1}e(0) \qquad (10.1\text{-}11)$$

$$e(0) = x(0) - \hat{x}(0) \qquad (10.1\text{-}12)$$

- In order for the error to approach zero asymptotically it is necessary that $\hat{A}$ be a stability matrix.
- Determination of the feedback matrix K is a pole-placement similar to response shaping of a system with full-state variable feedback.
- The eigenvalues of $\hat{A} = A - KC$ can be placed at arbitrary location if the observability test matrix is of rank k.

$$N = \begin{bmatrix} C' & A'C' & \cdots & (A')^{k-1}C' \end{bmatrix} \quad (10.1\text{-}12)$$

- If there is only a single output, then the observer gain matrix K becomes a column vector and is uniquely determined by the desired eigenvalues of $\hat{A}$.
- The presence of more than one output provides more flexibility: it is possible to place all the eigenvalues and do other things. Or, alternatively, some of the observer gains can be set to zero to simplify the resulting observer structure.

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky = \hat{A}\hat{x} + (B - KD)u + Ky \quad (10.1\text{-}13)$$

$$(sI - \hat{A})\hat{x}(s) = (B - KD)u(s) + Ky(s) + \hat{x}(0) \quad (10.1\text{-}14)$$

$$\hat{x}(s) = (sI - \hat{A})^{-1}(B - KD)u(s) + (sI - \hat{A})^{-1}Ky(s) + (sI - \hat{A})^{-1}\hat{x}(0) \quad (10.1\text{-}15)$$

where $\hat{A} = A - KC$.

10.2 Pole-Placement for Single-Output Systems

$$y = c_1x_1 + c_2x_2 + \cdots + c_kx_k = \begin{bmatrix} c_1 & c_2 & \cdots & c_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = Cx = c'x \qquad (10.2\text{-}1)$$

$$K = k = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_k \end{bmatrix} \qquad (10.2\text{-}2)$$

$$\hat{A} = A - kc' \qquad (10.2\text{-}3)$$

$$k = [(NW)']^{-1}(\hat{a} - a) \qquad (10.2\text{-}4)$$

$$N = \begin{bmatrix} C' & A'C' & \cdots & (A')^{k-1}C' \end{bmatrix} \qquad (10.2\text{-}5)$$

$$\hat{a} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_k \end{bmatrix}, a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} \qquad (10.2\text{-}6)$$

$\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_k$: the coefficients of the desired characteristic equation:

$$\left| sI - \hat{A} \right| = s^k + \hat{a}_1 s^{k-1} + \hat{a}_2 s^{k-2} + \cdots + \hat{a}_k = 0 \qquad (10.2\text{-}7)$$

$a_1, a_2, \cdots, a_k$: the coefficients of the original characteristic equation:

$$|sI - A| = s^k + a_1 s^{k-1} + a_2 s^{k-2} + \cdots + a_k = 0 \quad (10.2\text{-}8)$$

$$W = \begin{bmatrix} 1 & a_1 & \cdots & a_{k-1} \\ 0 & 1 & \cdots & a_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad (10.2\text{-}9)$$

Example: Consider a dc motor driving an inertia load.

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) \quad (10.2\text{-}10)$$

$$\hat{x} = \begin{bmatrix} \hat{e} \\ \hat{\omega} \end{bmatrix} \quad (10.2\text{-}11)$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}, B = \begin{bmatrix} 0 \\ \beta \end{bmatrix} \quad (10.2\text{-}12)$$

$$y = e = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e \\ \omega \end{bmatrix} = Cx \quad (10.2\text{-}13)$$

$$\begin{bmatrix} \dot{\hat{e}} \\ \dot{\hat{\omega}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} \hat{e} \\ \hat{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} [u] + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e \\ \omega \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{e} \\ \hat{\omega} \end{bmatrix} \right\} \quad (10.2\text{-}14)$$

$$\dot{\hat{e}} = \hat{\omega} + k_1(e - \hat{e}) \quad (10.2\text{-}15)$$

$$\dot{\hat{\omega}} = -\alpha\hat{\omega} + \beta u + k_2(e - \hat{e}) \quad (10.2\text{-}16)$$

$$K = k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \tag{10.2-17}$$

$$k = [(NW)']^{-1}(\hat{a} - a) \tag{10.2-18}$$

$$N = \begin{bmatrix} C' & A'C' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{10.2-19}$$

$$W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \tag{10.2-20}$$

$$(NW)' = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}, [(NW)']^{-1} = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix} \tag{10.2-21}$$

The open-loop characteristic polynomial,

$$D(s) = s^2 + \alpha s \tag{10.2-22}$$

$$a = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \tag{10.2-23}$$

The desired observer characteristic polynomial,

$$\hat{D}(s) = s^2 + \hat{a}_1 s + \hat{a}_2 \tag{10.2-24}$$

$$\hat{a} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} \tag{10.2-25}$$

$$k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_1 - \alpha \\ \hat{a}_2 \end{bmatrix} = \begin{bmatrix} \hat{a}_1 - \alpha \\ \hat{a}_2 - \alpha(\hat{a}_1 - \alpha) \end{bmatrix} \tag{10.2-26}$$

In the straight forward consideration,

$$\hat{A} = A - KC = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -k_1 & 1 \\ -k_2 & -\alpha \end{bmatrix} \tag{10.2-27}$$

$$(sI - \hat{A})^{-1} = \begin{bmatrix} s + k_1 & -1 \\ k_2 & s + \alpha \end{bmatrix}^{-1} = \frac{1}{(s + k_1)(s + \alpha) + k_2} \begin{bmatrix} s + \alpha & 1 \\ -k_2 & s + k_1 \end{bmatrix} \tag{10.2-28}$$

$$s^2 + (k_1 + \alpha)s + \alpha k_1 + k_2 = s^2 + \hat{a}_1 s + \hat{a}_2 \tag{10.2-29}$$

The relations between the estimation of state variables and the observation y and control input u,

$$\hat{x}(s) = (sI - \hat{A})^{-1}(B)u(s) + (sI - \hat{A})^{-1} K y(s) \tag{10.1-30}$$

$$(sI - \hat{A})^{-1} B = \frac{1}{s^2 + \hat{a}_1 s + \hat{a}_2} \begin{bmatrix} \beta \\ \beta(s + k_1) \end{bmatrix} \tag{10.2-31}$$

$$(sI - \hat{A})^{-1} K = \frac{1}{s^2 + \hat{a}_1 s + \hat{a}_2} \begin{bmatrix} (s + \alpha)k_1 + k_2 \\ k_2 s \end{bmatrix} = \frac{1}{s^2 + \hat{a}_1 s + \hat{a}_2} \begin{bmatrix} k_1 s + \hat{a}_2 \\ k_2 s \end{bmatrix} \tag{10.2-32}$$

$$\hat{x}_1(s) = \hat{e}(s) = \frac{\beta u(s) + (k_1 s + \hat{a}_2) e(s)}{s^2 + \hat{a}_1 s + \hat{a}_2} \tag{10.2-33}$$

$$\hat{x}_2(s) = \hat{\omega}(s) = \frac{\beta(s + k_1)u(s) + k_2 s e(s)}{s^2 + \hat{a}_1 s + \hat{a}_2} \tag{10.2-34}$$

- The control u that is used in the observer, as well as the input to the plant, is computed using the estimated state.

$$u = -G\hat{x} = -g_1\hat{e} - g_2\hat{\omega} \qquad (10.2\text{-}35)$$

10.3 Disturbances and Tracking Systems: Exogenous Variables

With the disturbance, x_d, and the reference input, x_r, $e = x - x_r$,

$$\dot{e} = Ae + Bu + Ex_0 \tag{10.3-1}$$

$$x_0 = \begin{bmatrix} x_r \\ \hline x_d \end{bmatrix} \tag{10.3-2}$$

$$\dot{x}_0 = A_0 x_0 \tag{10.3-3}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{10.3-4}$$

$$\mathbf{A} = \begin{bmatrix} A & | & E \\ \hline 0 & | & A_0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} B \\ \hline 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} e \\ x_0 \end{bmatrix} \tag{10.3-5}$$

The control law,

$$u = -Ge - G_0 x_0 \tag{10.3-6}$$

$$G_0 = \begin{bmatrix} G_r & G_d \end{bmatrix} \tag{10.3-7}$$

$$y = Ce + Dx_0 = \mathbf{C}\mathbf{x} \tag{10.3-8}$$

$$\mathbf{C} = \begin{bmatrix} C & D \end{bmatrix} \tag{10.3-9}$$

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{K}(y - \mathbf{C}\hat{\mathbf{x}}) \tag{10.3-10}$$

$$\begin{bmatrix} \dot{\hat{e}} \\ - \\ \dot{\hat{x}}_0 \end{bmatrix} = \begin{bmatrix} A & | & E \\ \text{—} & \text{—} & \text{—} \\ 0 & | & A_0 \end{bmatrix} \begin{bmatrix} \hat{e} \\ - \\ \hat{x}_0 \end{bmatrix} + \begin{bmatrix} B \\ \text{—} \\ 0 \end{bmatrix} [u] + \begin{bmatrix} K_e \\ - \\ K_0 \end{bmatrix} \left(y - [C \quad | \quad D] \begin{bmatrix} \hat{e} \\ - \\ \hat{x}_0 \end{bmatrix} \right) \tag{10.3-11}$$

$$\dot{\hat{e}} = A\hat{e} + Bu + E\hat{x}_0 + K_e(y - C\hat{e} - D\hat{x}_0) \tag{10.3-12}$$

$$\dot{\hat{x}}_0 = A_0\hat{x}_0 + K_0(y - C\hat{e} - D\hat{x}_0) \tag{10.3-13}$$

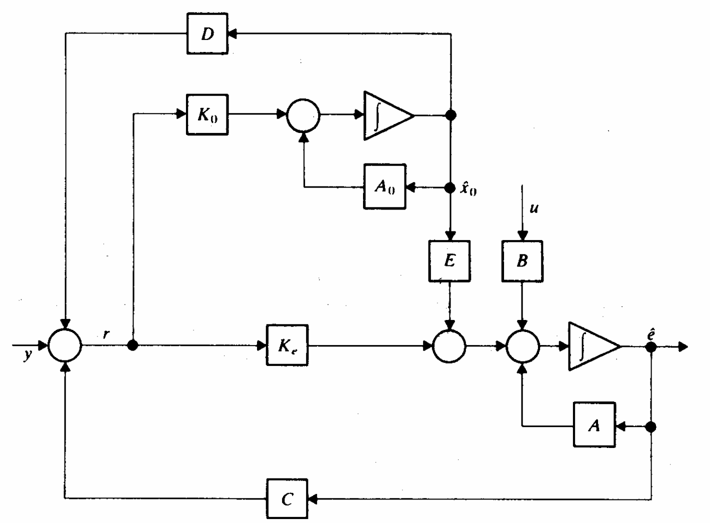


Figure 10.3-1 Block Diagram of Observer Including Estimation of Exogenous Vector

The closed-loop matrix for the metasystem,

$$\hat{\mathbf{A}} = \mathbf{A} - \mathbf{KC} = \begin{bmatrix} A - K_e C & E - K_e D \\ -K_0 C & A_0 - K_0 D \end{bmatrix} \tag{10.3-14}$$

- The closed-loop poles of the observer can be placed at arbitrary locations if the metasystem is observable, or **N** of the metasystem is invertible.

$$\mathbf{N} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' & \cdots & (\mathbf{A}')^{k+\nu-1}\mathbf{C} \end{bmatrix} \tag{10.3-15}$$

If the observer for the system error has already been designed, it might be desirable to amend the existing observer.

$$\hat{e} = \tilde{e} + V\hat{x}_0 \tag{10.3-16}$$

$\tilde{e}$: the observer for the process with $x_0 = 0$

$$\dot{\tilde{e}} = A\tilde{e} + Bu + \tilde{K}(y - C\tilde{e}) \tag{10.3-17}$$

$\tilde{K}$: the gain matrix for the observer in the absence of x_0

$$\begin{aligned} \dot{\hat{e}} &= \dot{\tilde{e}} + V\dot{\hat{x}}_0 \\ &= A\tilde{e} + Bu + \tilde{K}(y - C\tilde{e}) + V(A_0\hat{x}_0 + K_0(y - C(\tilde{e} + V\hat{x}_0) - D\hat{x}_0)) \\ &= A\tilde{e} + Bu + (\tilde{K} + VK_0)(y - C\tilde{e}) + (VA_0 - VK_0(CV + D))\hat{x}_0 \end{aligned} \tag{10.3-18}$$

$$\begin{aligned} \dot{\hat{e}} &= A(\tilde{e} + V\hat{x}_0) + Bu + E\hat{x}_0 + K_e(y - C\tilde{e} - (CV + D)\hat{x}_0) \\ &= A\tilde{e} + Bu + K_e(y - C\tilde{e}) + (AV + E - K_e(CV + D))\hat{x}_0 \end{aligned} \tag{10.3-19}$$

$$\tilde{K} + VK_0 = K_e \tag{10.3-20}$$

$$VA_0 - VK_0(CV + D) = AV + E - K_e(CV + D) \tag{10.3-21}$$

$$VA_0 - (A - \tilde{K}C)V = E - \tilde{K}D \tag{10.3-22}$$

$$V = [A_0 - (A - \tilde{K}C)]^{-1}(E - \tilde{K}D) \tag{10.3-23}$$

$$\dot{\hat{x}}_0 = A_0\hat{x}_0 + K_0(y - C(\tilde{e} + V\hat{x}_0) - D\hat{x}_0)$$

$$= (A_0 - K_0(CV + D))\hat{x}_0 + K_0(y - C\tilde{e}) = \tilde{A}_0\hat{x}_0 + K_0(y - C\tilde{e}) \tag{10.3-24}$$

- The input to the estimate of the exogenous vector is the residual of the observer for the process without exogenous inputs.

$$r = y - C\tilde{e} \tag{10.3-25}$$

- It is thus possible to design that observer first, and then to use its residual to drive the estimator of the exogenous inputs.
- In summary, the design of an observer of the system with exogeneous input:

Step 1. Design an observer (i.e., find the gain matrix $\tilde{K}$) for the process without exogenous inputs.

Step 2. Using the gain $\tilde{K}$ found in step 1, find the matrix V.

Step 3. Find K_0 so that dynamics matrix of the estimator of the exogenous vector has the desired pole locations.

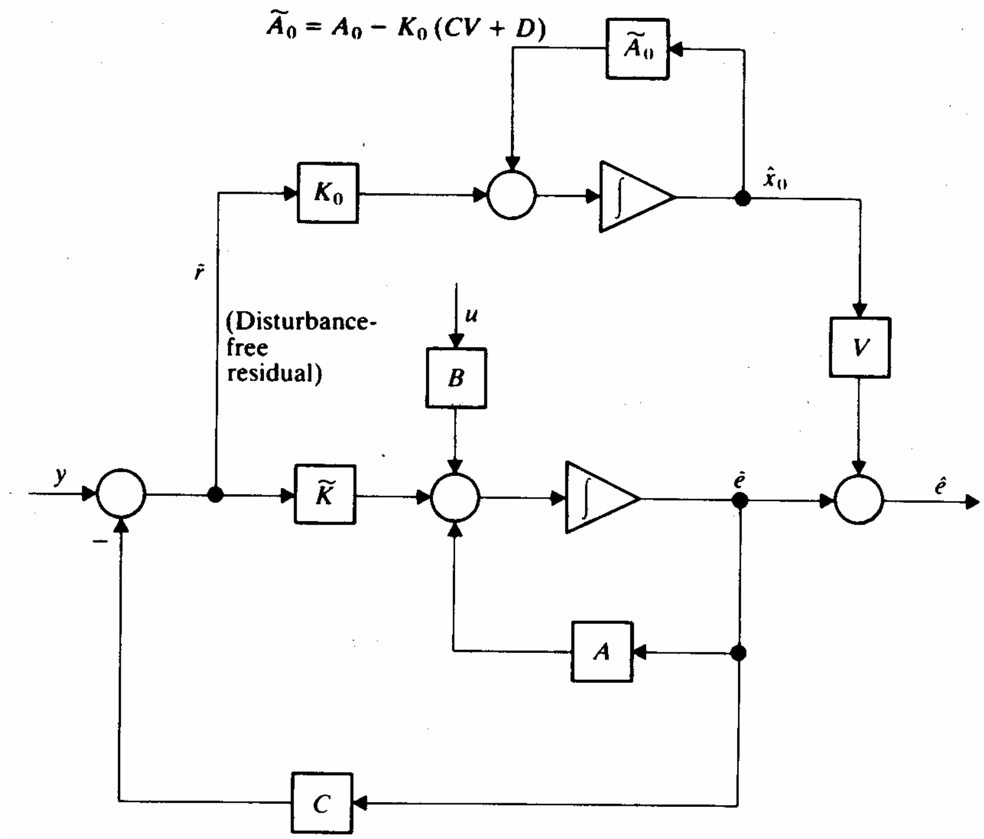


Figure 10.3-2 Alternate form of observer in which estimate of exogenous input is obtained using “disturbance-free” residual.

In the motor-driven inverted pendulum problem, if there is a constant disturbance, wind for instance, presented in addition to the control input, u, the complete dynamic model is

$$\dot{\theta} = \omega \tag{1}$$

$$\dot{\omega} = \Omega^2\theta - \alpha\omega + \beta u + d \tag{2}$$

$$\dot{d} = 0 \tag{3}$$

Thus,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \Omega^2 & -\alpha & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix} \tag{4}$$

or

$$E = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5}$$

If the observation vector depends only on the angular position θ, thus,

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ d \end{bmatrix} \tag{6}$$

Thus,

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ and } D = 0 \tag{7}$$

The observability test matrix for the metasystem is

$$\mathbf{N} = \begin{bmatrix}\mathbf{C}' & \mathbf{A}'\mathbf{C}' & \mathbf{A}'^2\mathbf{C}'\end{bmatrix} = \begin{bmatrix} 1 & 0 & \Omega^2 \\ 0 & 1 & -\alpha \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

The open-loop characteristic equation is

$$|sI - \mathbf{A}| = \begin{vmatrix} s & -1 & 0 \\ -\Omega^2 & s+\alpha & -1 \\ 0 & 0 & s \end{vmatrix} = s(s^2 + \alpha s - \Omega^2) \tag{9}$$

Thus,

$$a_1 = \alpha,\ a_2 = -\Omega^2, \text{ and } a_3 = 0 \tag{10}$$

and hence

$$\mathbf{W} = \begin{bmatrix} 1 & \alpha & -\Omega^2 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

Thus,

$$[(\mathbf{NW})^{-1}]' = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

When the desired characteristic equation of the observer is $s^3 + \hat{a}_1 s^2 + \hat{a}_2 s + \hat{a}_3 = 0$, the gain matrix is

$$\mathbf{K} = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_1 - \alpha \\ \hat{a}_2 + \Omega^2 \\ \hat{a}_3 \end{bmatrix} = \begin{bmatrix} \hat{a}_1 - \alpha \\ \hat{a}_2 + \Omega^2 - \alpha(\hat{a}_1 - \alpha) \\ \hat{a}_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \tag{13}$$

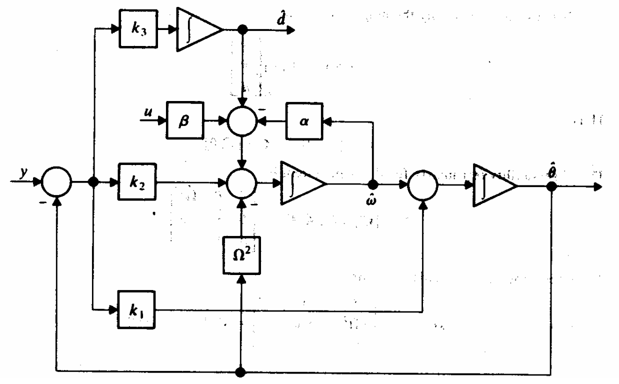
The observer dynamics are given by $\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{K}(y - \mathbf{C}\hat{\mathbf{x}})$,

$$\dot{\hat{\theta}} = \hat{\omega} + k_1(y - \hat{\theta}) \tag{14}$$

$$\dot{\hat{\omega}} = \Omega^2\hat{\theta} - \alpha\hat{\omega} + \beta u + \hat{d} + k_2(y - \hat{\theta}) \tag{15}$$

$$\dot{\hat{d}} = k_3(y - \hat{\theta}) \tag{16}$$

and has the block-diagram representation shown below.



For the alternate method of design, the disturbance-free observer follows $\dot{\tilde{e}} = A\tilde{e} + Bu + \tilde{K}(y - C\tilde{e})$.

Characteristic equation of the disturbance-free observer,

$$|sI - \mathbf{A}| = \begin{vmatrix} s & -1 \\ -\Omega^2 & s+\alpha \end{vmatrix} = s^2 + \alpha s - \Omega^2 = 0 \tag{17}$$

$$N = [C' \quad A'C'] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \tag{18}$$

Thus,

$$[(NW)^{-1}]' = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix} \tag{19}$$

When the desired characteristic equation of the disturbance-free observer is $s^2 + \hat{a}_1 s + \hat{a}_2 = 0$, the gain matrix is

$$\tilde{K} = [(NW)^{-1}]' \begin{bmatrix} \hat{a}_1 - a_1 \\ \hat{a}_2 - a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_1 - \alpha \\ \hat{a}_2 + \Omega^2 \end{bmatrix} = \begin{bmatrix} \hat{a}_1 - \alpha \\ \hat{a}_2 + \Omega^2 - \alpha(\hat{a}_1 - \alpha) \end{bmatrix} = \begin{bmatrix} \tilde{k}_1 \\ \tilde{k}_2 \end{bmatrix} \tag{20}$$

The closed-loop matrix of the disturbance-free observer is

$$\tilde{A} = A - \tilde{K}C = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix} - \begin{bmatrix} \tilde{k}_1 \\ \tilde{k}_2 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} -\tilde{k}_1 & 1 \\ \Omega^2 - \tilde{k}_2 & -\alpha \end{bmatrix} \tag{21}$$

The correction matrix, $V = -(A - \tilde{K}C)^{-1}E = -\tilde{A}^{-1}E$

$$\tilde{A}^{-1} = (A - \tilde{K}C)^{-1} = \frac{1}{\hat{a}_2} \begin{bmatrix} -\alpha & -1 \\ -\Omega^2 + \tilde{k}_2 & -\tilde{k}_1 \end{bmatrix} \tag{22}$$

$$V = -(A - \tilde{K}C)^{-1}E = \frac{1}{\hat{a}_2}\begin{bmatrix} \alpha & 1 \\ \Omega^2 - \tilde{k}_2 & \tilde{k}_1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\hat{a}_2}\begin{bmatrix} 1 \\ \tilde{k}_1 \end{bmatrix} \quad (23)$$

The disturbance estimator follows $\dot{\hat{x}}_0 = (A_0 - K_0(CV + D))\hat{x}_0 + K_0(y - C\tilde{e}) = \tilde{A}_0\hat{x}_0 + K_0(y - C\tilde{e})$.

$$CV + D = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\hat{a}_2 \\ \tilde{k}_1/\hat{a}_2 \end{bmatrix} = \frac{1}{\hat{a}_2} \quad (24)$$

$$\dot{\hat{d}} = -(k_d/\hat{a}_2)\hat{d} + k_d(y - \tilde{\theta}) \quad (25)$$

The disturbance-estimation gain k_d can determined from the remaining of the desired characteristic equation.

$$\left| sI - \tilde{A}_0 \right| = s + k_d/\hat{a}_2 = s + \hat{a}_d \quad (26)$$

$$k_d = \hat{a}_2\hat{a}_d \quad (27)$$

The equation for the disturbance-free observer is

$$\dot{\tilde{\theta}} = \tilde{\omega} + \tilde{k}_1(y - \tilde{\theta}) \quad (28)$$

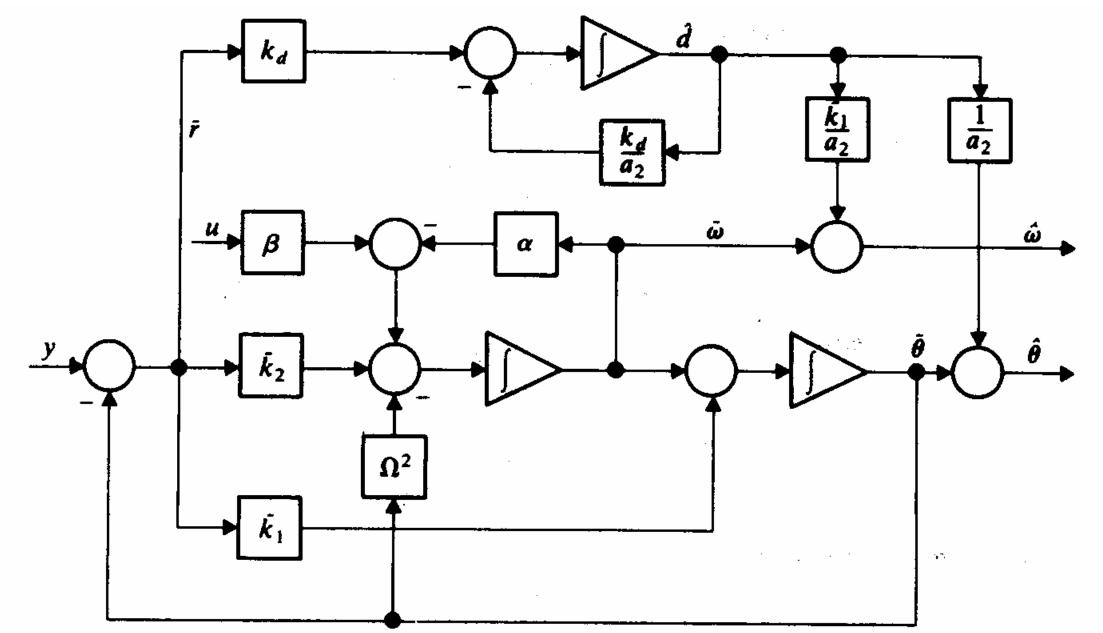
$$\dot{\tilde{\omega}} = \Omega^2\tilde{\theta} - \alpha\tilde{\omega} + \beta u + \tilde{k}_2(y - \tilde{\theta}) \quad (29)$$

when $\hat{e} = \tilde{e} + V\hat{x}_0$,

$$\hat{\theta} = \tilde{\theta} + \frac{1}{\hat{a}_2}\hat{d} \quad (30)$$

$$\hat{\omega} = \tilde{\omega} + \frac{\tilde{k}_1}{\hat{a}_2}\hat{d} \quad (31)$$

A block diagram showing the implementation is given below.



The overall dynamic of the observer follows $(s^2 + \hat{a}_1 s + \hat{a}_2)(s + \hat{a}_d) = 0$.

10.4 Reduced-Order Observers

$$y = Cx \tag{10.4-1}$$

When C is a nonsingular matrix,

$$x = \hat{x} = C^{-1}y \tag{10.4-2}$$

- In many applications, it is possible to group the state variables into two sets: those that can be measured directly and those that depend indirectly on the former.

$$x = \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix} \tag{10.4-3}$$

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u \tag{10.4-4}$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u \tag{10.4-5}$$

$$y = C_1x_1 = \begin{bmatrix} C_1 & | & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix} \tag{10.4-6}$$

$$\dot{\hat{x}}_1 = A_{11}\hat{x}_1 + A_{12}\hat{x}_2 + B_1u + K_1(y - C_1\hat{x}_1) \tag{10.4-7}$$

$$\dot{\hat{x}}_2 = A_{21}\hat{x}_1 + A_{22}\hat{x}_2 + B_2u + K_2(y - C_1\hat{x}_1) \tag{10.4-8}$$

$$x_1 = \hat{x}_1 = C_1^{-1} y \tag{10.4-9}$$

$$\dot{\hat{x}}_2 = A_{21} C_1^{-1} y + A_{22} \hat{x}_2 + B_2 u \tag{10.4-10}$$

- The dynamic behavior of the reduced-order observer is governed by the eigenvalues of A_{22} which is a submatrix of the open-loop dynamics matrix A, a matrix over which the designer has no control.
- If the eigenvalues of A_{22} are suitable, then the observer could be designed. But there is no assurance that the eigenvalues of A_{22} are suitable.

A suitably general structure for the estimation of $\hat{x}_{22}$ is given by

$$\hat{x}_2 = Ly + z \tag{10.4-11}$$

$$\dot{z} = Fz + \overline{G} y + Hu \tag{10.4-12}$$

$$e = x - \hat{x} = \begin{bmatrix} x_1 - \hat{x}_1 \\ \hline x_2 - \hat{x}_2 \end{bmatrix} = \begin{bmatrix} e_1 \\ \hline e_2 \end{bmatrix} \tag{10.4-13}$$

$$e_1 = x_1 - \hat{x}_1 = 0 \tag{10.4-14}$$

$$\begin{aligned} \dot{e}_2 &= \dot{x}_2 - \dot{\hat{x}}_2 = A_{21} x_1 + A_{22} x_2 + B_2 u - L\dot{y} - \dot{z} \\ &= A_{21} x_1 + A_{22} x_2 + B_2 u - L[C_1 (A_{11} x_1 + A_{12} x_2 + B_1 u)] - Fz - \overline{G} y - Hu \end{aligned} \tag{10.4-15}$$

$$z = \hat{x}_2 - Ly = x_2 - e_2 - Ly = x_2 - e_2 - LC_1 x_1 \tag{10.4-16}$$

$$\dot{e}_2 = Fe_2 + (A_{21} - LC_1A_{11} - \overline{G}C_1 + FLC_1)x_1 + (A_{22} - LC_1A_{12} - F)x_2 + (B_2 - LC_1B_1 - H)u \quad (10.4\text{-}17)$$

$$F = A_{22} - LC_1A_{12} \quad (10.4\text{-}18)$$

$$H = B_2 - LC_1B_1 \quad (10.4\text{-}19)$$

$$\overline{G}C_1 = A_{21} - LC_1A_{11} + FLC_1 \quad (10.4\text{-}20)$$

$$\dot{e}_2 = Fe_2 \quad (10.4\text{-}21)$$

$$e_2(s) = (sI - F)^{-1}e_2(0) \quad (10.4\text{-}22)$$

- Selecting the gain matrix L of the reduced order observer to place the eigenvalues of F is the same type of problem as selecting the gain matrix K to place the eigenvalues of $\hat{A}$.
- In order to place the poles of F, it is necessary that the corresponding controllability test matrix is invertible.

$$N_1 = \begin{bmatrix} A'_{12}C'_1 & A'_{22}A'_{12}C'_1 & \cdots & (A'_{22})^{k-l-1}A'_{12}C'_1 \end{bmatrix} \quad (10.4\text{-}23)$$

$$\overline{G} = (A_{21} - LC_1A_{11})C_1^{-1} + FL \quad (10.4\text{-}24)$$

$$\dot{z} = Fz + \overline{G}y + Hu = F(\hat{x}_2 - Ly) + [(A_{21} - LC_1A_{11})C_1^{-1} + FL]y + Hu = F\hat{x}_2 + (A_{21} - LC_1A_{11})C_1^{-1}y + Hu = F\hat{x}_2 + \overline{g}y + Hu \quad (10.4\text{-}25)$$

$$\overline{g} = (A_{21} - LC_1A_{11})C_1^{-1} \quad (10.4\text{-}26)$$

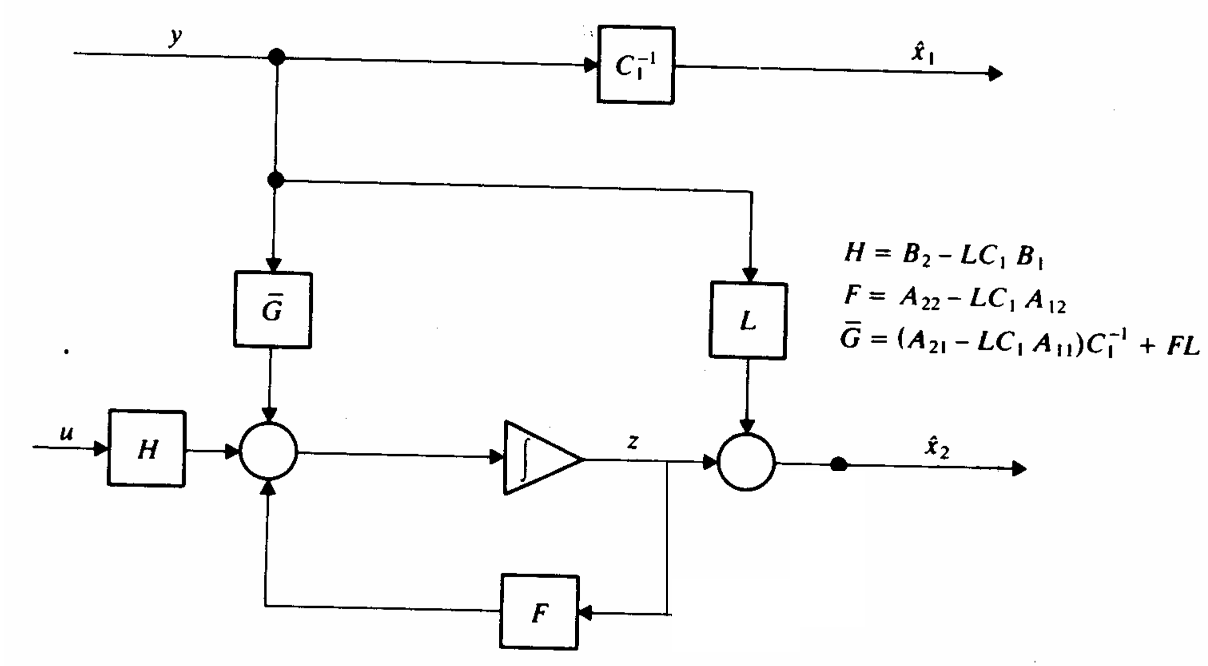


Figure 10.4-1 Reduced-Order Observer for Observation $y = C_1x_1$ with C_1 Nonsingular

Example: Metastate form of a system is represented by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_d \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & e_1 \\ a_{21} & a_{22} & e_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_d \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} u \qquad (10.4\text{-}27)$$

A. Full-order observer, assuming only x_1 is measured.

$$y = x_1 \text{ or } C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad (10.4\text{-}28)$$

B. Full-order observer, assuming both x_1 and x_2 are measured.

$$y_1 = x_1 \text{ and } y_2 = x_2 \text{ or } C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (10.4\text{-}29)$$

C. Reduced- (second-) order observer, assuming only x_1 is measured.

D. Reduced- (first-) order observer, assuming both x_1 and x_2 are measured.

Case A Full-order observer with one measured variable

$$\dot{\hat{x}}_1 = a_{11}\hat{x}_1 + a_{12}\hat{x}_2 + e_1\hat{x}_d + b_1 u + k_1(y - \hat{x}_1) \tag{10.4-30}$$

$$\dot{\hat{x}}_2 = a_{21}\hat{x}_1 + a_{22}\hat{x}_2 + e_2\hat{x}_d + b_2 u + k_2(y - \hat{x}_1) \tag{10.4-31}$$

$$\dot{\hat{x}}_d = k_3(y - \hat{x}_1) \tag{10.4-32}$$

The gain matrix

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \tag{10.4-33}$$

The open-loop characteristic polynomial

$$|sI - A| = \begin{vmatrix} s - a_{11} & -a_{12} & -e_1 \\ -a_{21} & s - a_{22} & -e_2 \\ 0 & 0 & s \end{vmatrix} = s[s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}] \tag{10.4-34}$$

$$a_1 = -(a_{11} + a_{22}),\ a_2 = a_{11}a_{22} - a_{12}a_{21},\ \text{and}\ a_3 = 0 \tag{10.4-35}$$

$$W = \begin{bmatrix} 1 & -(a_{11} + a_{22}) & a_{11}a_{22} - a_{12}a_{21} \\ 0 & 1 & -(a_{11} + a_{22}) \\ 0 & 0 & 1 \end{bmatrix} \tag{10.4-36}$$

The observability test matrix

$$N = \begin{bmatrix} C' & A'C' & (A')^2 C' \end{bmatrix} = \begin{bmatrix} 1 & a_{11} & a_{11}^2 + a_{12}a_{21} \\ 0 & a_{12} & a_{11}a_{12} + a_{12}a_{22} \\ 0 & e_1 & a_{11}e_1 + a_{12}e_2 \end{bmatrix} \tag{10.4-37}$$

$$NW = \begin{bmatrix} 1 & -a_{22} & 0 \\ 0 & a_{12} & 0 \\ 0 & e_1 & -a_{22}e_1 + a_{12}e_2 \end{bmatrix} \tag{10.4-38}$$

$$K = [(NW)^{-1}]' \begin{bmatrix} \hat{a}_1 + (a_{11} + a_{22}) \\ \hat{a}_2 - a_{11}a_{22} + a_{12}a_{21} \\ \hat{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_{22}/a_{12} & 1/a_{12} & -e_1/[a_{12}(-a_{22}e_1 + a_{12}e_2)] \\ 0 & 0 & 1/(-a_{22}e_1 + a_{12}e_2) \end{bmatrix} \begin{bmatrix} \hat{a}_1 + (a_{11} + a_{22}) \\ \hat{a}_2 - a_{11}a_{22} + a_{12}a_{21} \\ \hat{a}_3 \end{bmatrix} \tag{10.4-39}$$

Case B Full-order observer with two measured variables

$$\dot{\hat{x}}_1 = a_{11}\hat{x}_1 + a_{12}\hat{x}_2 + e_1\hat{x}_d + b_1 u + k_{11}(y_1 - \hat{x}_1) + k_{12}(y_2 - \hat{x}_2) \quad (10.4\text{-}40)$$

$$\dot{\hat{x}}_2 = a_{21}\hat{x}_1 + a_{22}\hat{x}_2 + e_2\hat{x}_d + b_2 u + k_{21}(y_1 - \hat{x}_1) + k_{22}(y_2 - \hat{x}_2) \quad (10.4\text{-}41)$$

$$\dot{\hat{x}}_d = k_{31}(y_1 - \hat{x}_1) + k_{32}(y_2 - \hat{x}_2) \quad (10.4\text{-}42)$$

The observer gain matrix

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ k_{31} & k_{32} \end{bmatrix} \quad (10.4\text{-}43)$$

- There are six gains to be selected: twice as many as are needed to place all the closed-loop poles. There are many solutions.
- If y_1 is used to estimate x_1 and y_2 is used to estimate x_2. For estimating x_d, we might consider using the sum of $y_1 - \hat{x}_1$ and $y_2 - \hat{x}_2$, which would happen when $k_{31} = k_{32} = k_3$.

$$K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \\ k_3 & k_3 \end{bmatrix} \quad (10.4\text{-}44)$$

- Determination of the three gains k_1, k_2, and k_3 needed to place the eigenvalues of $\hat{A} = A - KC$ is straightforward problem in algebra.

Case C Reduced-order observer with one measurement

$$x = \begin{bmatrix} x_1 \\ - \\ x_2 \\ x_d \end{bmatrix}, A = \begin{bmatrix} a_{11} & | & a_{12} & e_1 \\ - & | & - & - \\ a_{21} & | & a_{22} & e_2 \\ 0 & | & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ - \\ b_2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & | & 0 & 0 \end{bmatrix} \tag{10.4-45}$$

$$\hat{x}_1 = C_1^{-1} y = y \tag{10.4-46}$$

$$\hat{x}_2 = Ly + z \tag{10.4-47}$$

$$\hat{x}_2 = l_1 y + z_1 \tag{10.4-48}$$

$$\hat{x}_d = l_2 y + z_2 \tag{10.4-49}$$

$$\dot{z} = F\hat{x}_2 + \bar{g}y + Hu \tag{10.4-50}$$

$$\dot{z}_1 = f_{11}\hat{x}_2 + f_{12}\hat{x}_d + \bar{g}_1 y + h_1 u \tag{10.4-51}$$

$$\dot{z}_2 = f_{21}\hat{x}_2 + f_{22}\hat{x}_d + \bar{g}_2 y + h_2 u \tag{10.4-52}$$

$$F = A_{22} - LC_1 A_{12} = \begin{bmatrix} a_{22} & e_2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} a_{12} & e_1 \end{bmatrix} = \begin{bmatrix} a_{22} - l_1 a_{12} & e_2 - l_1 e_1 \\ -l_2 a_{12} & -l_2 e_1 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \tag{10.4-53}$$

$$\bar{g} = (A_{21} - LC_1 A_{11}) C_1^{-1} = \begin{bmatrix} a_{21} \\ 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [a_{11}] = \begin{bmatrix} a_{21} - l_1 a_{11} \\ -l_2 a_{11} \end{bmatrix} = \begin{bmatrix} \bar{g}_1 \\ \bar{g}_2 \end{bmatrix} \tag{10.4-54}$$

$$H = B_2 - LC_1 B_1 = \begin{bmatrix} b_2 \\ 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [b_1] = \begin{bmatrix} b_2 - l_1 b_1 \\ -l_2 b_1 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \tag{10.4-55}$$

The characteristic equation for F

$$|sI - F| = s^2 - (f_{11} + f_{22})s + f_{11}f_{22} - f_{12}f_{21} = s^2 + \hat{a}_1 s + \hat{a}_2 = 0 \tag{10.4-56}$$

$$\hat{a}_1 = -a_{22} + l_1 a_{12} + l_2 e_1 \tag{10.4-57}$$

$$\hat{a}_2 = -a_{22} e_1 l_2 + a_{12} e_2 l_2 \tag{10.4-58}$$

$$l_2 = \frac{\hat{a}_2}{-a_{22}e_1 + a_{12}e_2} \tag{10.4-59}$$

$$l_1 = \frac{1}{a_{12}}\left[\hat{a}_1 + a_{22} - \frac{\hat{a}_2 e_1}{-a_{22}e_1 + a_{12}e_2}\right] \tag{10.4-60}$$

Case D Reduced-order observer with two measurements

$$x = \begin{bmatrix} x_1 \\ x_2 \\ - \\ x_d \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & | & e_1 \\ a_{21} & a_{22} & | & e_2 \\ - & - & | & - \\ 0 & 0 & | & 0 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ - \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix} \qquad (10.4\text{-}61)$$

$$\hat{x}_1 = C_1^{-1} y = y \qquad (10.4\text{-}62)$$

$$\hat{x}_1 = y_1 \qquad (10.4\text{-}63)$$

$$\hat{x}_2 = y_2 \qquad (10.4\text{-}64)$$

$$\hat{x}_2 = Ly + z \qquad (10.4\text{-}65)$$

$$\hat{x}_d = l_1 y_1 + l_2 y_2 + z \qquad (10.4\text{-}66)$$

$$\dot{z} = F\hat{x}_2 + \overline{g}y + Hu \qquad (10.4\text{-}67)$$

$$\dot{z} = f\hat{x}_d + \overline{g}_1 y_1 + \overline{g}_2 y_2 + hu \qquad (10.4\text{-}68)$$

$$F = A_{22} - LC_1 A_{12} = [0] - [l_1 \quad l_2] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = -l_1 e_1 - l_2 e_2 \qquad (10.4\text{-}69)$$

$$\overline{g} = (A_{21} - LC_1 A_{11}) C_1^{-1} = [0 \quad 0] - [l_1 \quad l_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = [-l_1 a_{11} - l_2 a_{21} \quad -l_1 a_{12} - l_2 a_{22}] \qquad (10.4\text{-}70)$$

$$H = B_2 - LC_1 B_1 = [0] - [l_1 \quad l_2] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = -l_1 b_1 - l_2 b_2 \qquad (10.4\text{-}71)$$

The characteristic equation for F

$$\left|sI - F\right| = s + l_1 e_1 + l_2 e_2 = s + \hat{a}_1 = 0 \tag{10.4-72}$$

11 Compensator Design by the Separation Principle

11.1 Compensators Designed Using Full-Order Observers

The standard dynamic process

$$\dot{x} = Ax + Bu \tag{11.1-1}$$

Observation

$$y = Cx \tag{11.1-2}$$

A full-state feedback control law

$$u = -Gx \tag{11.1-3}$$

An observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) \tag{11.1-4}$$

The control law in the separation principle

$$u = -G\hat{x} \tag{11.1-5}$$

The combined plant dynamics

$$\dot{x} = Ax - BG\hat{x} \tag{11.1-6}$$

The combined observer

$$\dot{\hat{x}} = A\hat{x} - BG\hat{x} + K(Cx - C\hat{x}) \tag{11.1-7}$$

The observer error

$$e = x - \hat{x} \tag{11.1-8}$$

$$\dot{x} = Ax - BG(x - e) = (A - BG)x + BGe = A_c x + BGe \tag{11.1-9}$$

$$\dot{e} = (A - KC)e = \hat{A}e \tag{11.1-10}$$

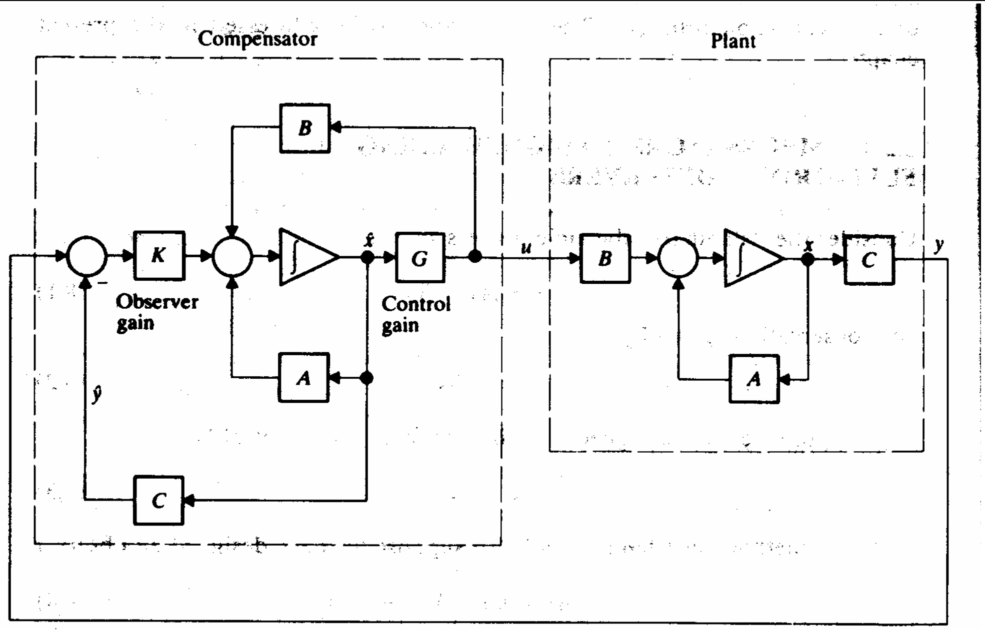


Figure 11.1 Control System Using Observer in Compensator

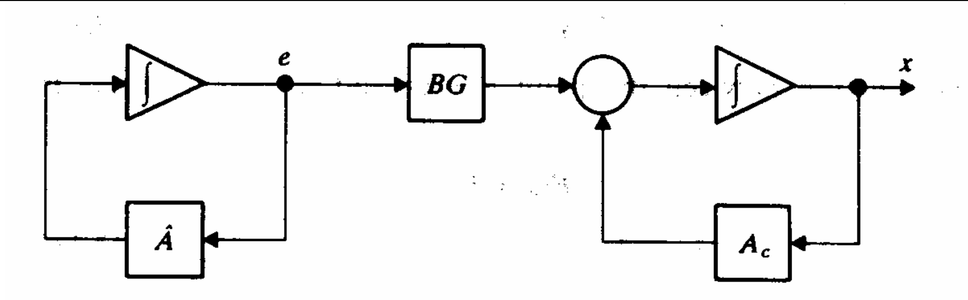


Figure 11.1-2 Block-Diagram Representation of State and Error in System with Compensator Designed by Separation Principle

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_c & BG \\ 0 & \hat{A} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} = \hat{A}_c \begin{bmatrix} x \\ e \end{bmatrix} \quad (11.1\text{-}11)$$

The combined plant characteristic equation

$$\left| sI - \hat{A}_c \right| = \begin{vmatrix} sI - A_c & -BG \\ 0 & sI - \hat{A} \end{vmatrix} = \left| sI - A_c \right| \left| sI - \hat{A} \right| = 0 \quad (11.1\text{-}12)$$

$$(sI - \hat{A})e(s) = e_0 \quad (11.1\text{-}13)$$

$$e(s) = (sI - \hat{A})^{-1} e_0 \quad (11.1\text{-}14)$$

$$(sI - A_c)x(s) = BGe(s) + x_0 \quad (11.1\text{-}15)$$

$$x(s) = (sI - A_c)^{-1} BGe(s) + (sI - A_c)^{-1} x_0 = (sI - A_c)^{-1} BG(sI - \hat{A})^{-1} e_0 + (sI - A_c)^{-1} x_0 \quad (11.1\text{-}16)$$

$$(sI - A_c)^{-1} BG (sI - \hat{A})^{-1} = \frac{\text{adj}(sI - A_c) BG \, \text{adj}(sI - \hat{A})}{|sI - A_c||sI - \hat{A}|} \tag{11.1-17}$$

$$\dot{\hat{x}} = A\hat{x} - BG\hat{x} + K(Cx - C\hat{x}) = (A - BG - KC)\hat{x} + Ky \tag{11.1-18}$$

$$\hat{x}(s) = (sI - A + BG + KC)^{-1} K y(s) \tag{11.1-19}$$

$$u(s) = -G\hat{x}(s) = -G(sI - A + BG + KC)^{-1} K y(s) \tag{11.1-20}$$

The transfer function $D(s)$ of the compensator

$$u(s) = -D(s) y(s) \tag{11.1-21}$$

$$D(s) = G(sI - A + BG + KC)^{-1} K = G(sI - \hat{A}_c)^{-1} K \tag{11.1-22}$$

$$\hat{A}_c = A - BG - KC = \hat{A} - BG = A_c - KC \tag{11.1-23}$$

The steps of the compensator design using observers:

Step 1. Design the control law under the assumption that all state variables in the process can be measured.

Step 2. Design an observer to estimate the state of the process for which the control law of step 1 was designed.

Step 3. Combine the full-state control law design of step 1 with the observer design of step 2 to obtain the compensator design.

Compensator designed using full-order observer is sampled in the motor-driven inverted pendulum problem here.

Step 1. Full-state feedback design

The other design of the control system with constant disturbance is shown here. The dynamics, including the disturbance are given by

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [d] \tag{1}$$

The control law for this process is

$$u = -Gx - G_0 x_0 \tag{2}$$

The gain matrix G was obtained before and it is used here. For the desired free-disturbance characteristic of $s^2 + \bar{a}_1 s + \bar{a}_2$,

$$G = g' = \begin{bmatrix} (\bar{a}_2 + \Omega^2)/\beta \\ (\bar{a}_1 - \alpha)/\beta \end{bmatrix}' \tag{3}$$

In addition, we need the disturbance gain g_0 which is computed by

$$g_0 = B^{\#} E \tag{4}$$

where E is the matrix that multiplies the disturbance, in this case

$$E = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5}$$

and since

$$B^{\#} = (CA_c^{-1}B)^{-1}CA_c^{-1} \tag{6}$$

where

$$A_c = A - BG = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix} - \begin{bmatrix} 0 \\ \beta \end{bmatrix} \begin{bmatrix} (\bar{a}_2 + \Omega^2)/\beta & (\bar{a}_2 - \alpha)/\beta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\bar{a}_2 & -\bar{a}_1 \end{bmatrix} \tag{7}$$

The observation matrix C is needed for the computation of $B^{\#}$. For this example, we assume that our sole measurement is of the pendulum angular position θ, that is

$$y = Cx \tag{8}$$

with

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{9}$$

Thus

$$CA_c^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -\bar{a}_1 & -1 \\ \bar{a}_2 & 0 \end{bmatrix} \frac{1}{\bar{a}_2} = \begin{bmatrix} -\bar{a}_1/\bar{a}_2 & -1/\bar{a}_2 \end{bmatrix} \tag{10}$$

Hence, with

$$B = \begin{bmatrix} 0 \\ \beta \end{bmatrix} \tag{11}$$

$$B^{\#} = \begin{bmatrix} \bar{a}_1/\beta & 1/\beta \end{bmatrix} \tag{12}$$

Thus from (4),

$$g_0 = \frac{1}{\beta} \tag{13}$$

Thus the full-state feedback control law is

$$u = -\left(\frac{\bar{a}_2 + \Omega^2}{\beta}\right)\theta - \left(\frac{\bar{a}_1 - \alpha}{\beta}\right)\omega - \frac{1}{\beta}d \tag{14}$$

Step 2. Observer design with known control

The observer designed under the assumption that the control is known was derived before, it is used here.

$$\dot{\hat{\theta}} = \hat{\omega} + k_1(y - \hat{\theta}) \tag{15}$$

$$\dot{\hat{\omega}} = \Omega^2\hat{\theta} - \alpha\hat{\omega} + \beta u + \hat{d} + k_2(y - \hat{\theta}) \tag{16}$$

$$\dot{\hat{d}} = k_3(y - \hat{\theta}) \tag{17}$$

For the desired observer characteristic of $s^3 + \hat{a}_1 s^2 + \hat{a}_2 s + \hat{a}_3$, the observer gain matrix given by

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} \hat{a}_1 - \alpha \\ \hat{a}_2 + \Omega^2 - \alpha(\hat{a}_1 - \alpha) \\ \hat{a}_3 \end{bmatrix} \tag{18}$$

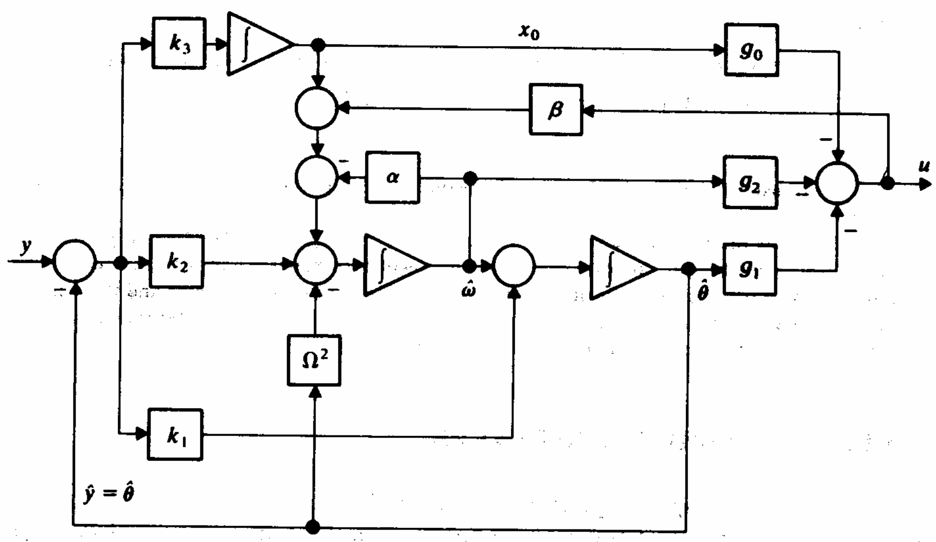
Step 3. Compensator design

The compensator dynamic equations are obtained by using the estimated state in (14).

$$u = -\left(\frac{\bar{a}_2 + \Omega^2}{\beta}\right)\hat{\theta} - \left(\frac{\bar{a}_1 - \alpha}{\beta}\right)\hat{\omega} - \frac{1}{\beta}\hat{d} \tag{19}$$

and also using this control in (15)-(17).

A block-diagram representation of (15)-(17) and (19) is shown in figure below, which is the same as the block diagram for the observer with known input, but with the input u given by (19).



Although the structure of the figure explicitly exhibits the estimates of the state variables, it is not necessary that the compensator be implemented by that structure. As long as the transfer function between the measure state $y = \theta$ and the control output u is the same as the transfer function between y and u in the figure, the closed-loop system will have the same behavior.

The compensator transfer function is

$$D(s) = G(sI - \hat{A}_c)^{-1} K \tag{20}$$

where

$$\hat{A}_c = \hat{A} - BG = \begin{bmatrix} -k_1 & 1 & 0 \\ -k_2 + \Omega^2 & -\alpha & 1 \\ -k_3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix} \begin{bmatrix} g & g_0 \end{bmatrix} = \begin{bmatrix} -\hat{a}_1 + \alpha & 1 & 0 \\ -\hat{a}_2 + \alpha(\hat{a}_1 - \alpha) - \bar{a}_2 - \Omega^2 & -\bar{a}_1 & 0 \\ -\hat{a}_3 & 0 & 0 \end{bmatrix} \tag{21}$$

The resolvent for $\hat{A}_c$ is given by

$$(sI - \hat{A}_c)^{-1} = \frac{1}{\Delta} \begin{bmatrix} s(s - \bar{a}_1) & s & 0 \\ -s[\hat{a}_2 + \bar{a}_2 + \Omega^2 \alpha(\hat{a}_1 - \alpha)] & s(s + \hat{a}_1 - \alpha) & 0 \\ -\hat{a}_3(s + \hat{a}_1) & -\hat{a}_3 & s^2 + (\hat{a}_1 + \bar{a}_1 - \alpha)s + (\hat{a}_1 - \alpha)(\bar{a}_1 - \alpha) + \hat{a}_2 + \bar{a}_2 + \Omega^2 \end{bmatrix} \tag{22}$$

where

$$\Delta = s[s^2 + (\hat{a}_1 + \bar{a}_1 - \alpha)s + (\hat{a}_1 - \alpha)(\bar{a}_1 - \alpha) + \hat{a}_2 + \bar{a}_2 + \Omega^2] \tag{23}$$

After some calculation the transfer function of the compensator is determined to be

$$D(s) = \frac{d_1 s^2 + d_2 s + d_3}{\beta s[s^2 + (\hat{a}_1 + \bar{a}_1 - \alpha)s + (\hat{a}_1 - \alpha)(\bar{a}_1 - \alpha) + \hat{a}_2 + \bar{a}_2 + \Omega^2} \quad (24)$$

where

$$d_1 = \bar{a}_2(\hat{a}_1 - \alpha) + \hat{a}_2(\bar{a}_1 - \alpha) - \alpha(\hat{a}_1 - \alpha)(\bar{a}_1 - \alpha) + \Omega^2(\hat{a}_1 + \bar{a}_1 - 2\alpha) + \hat{a}_3 \quad (25)$$

$$d_2 = \hat{a}_2\bar{a}_2 + \Omega^2[(\hat{a}_1 - \alpha)(\bar{a}_1 - \alpha) + \hat{a}_2] + \Omega^4 + \hat{a}_1\hat{a}_3 \quad (26)$$

$$d_3 = \hat{a}_3[\bar{a}_2 + (\hat{a}_1 - \alpha)(\bar{a}_1 - \alpha)] \quad (27)$$

Note that the transfer function of the compensator as given by $D(s)$ has a pole at the origin which resulted in this case form the unknown disturbance which is estimated by the observer. As a result of the pole at the origin, the cascade of the compensator and the original plant aslo has a pole at the origin, resulting in a “type 1” closed-loop transfer function which will ensure that the steady-state error for a constant disturbance is zero.

11.2 Reduced-Order Observers

For the special case in which the observation can be used to solve for a substate:

$$y = C_1 x_1 \tag{11.2-1}$$

with C_1 being a nonsingular matrix,

$$\hat{x}_1 = x_1 = C_1^{-1} y \tag{11.2-2}$$

$$\hat{x}_2 = Ly + z \tag{11.2-3}$$

$$\dot{z} = Fz + \overline{G}y + Hu = F\hat{x}_2 + \overline{g}y + Hu \tag{11.2-4}$$

$$F = A_{22} - LC_1 A_{12} \tag{11.2-5}$$

$$\overline{G} = A_{21} - LC_1 A_{11} + FL; \overline{g} = (A_{21} - LC_1 A_{11})C_1^{-1} \tag{11.2-6}$$

$$H = B_2 - LC_1 B_1 \tag{11.2-7}$$

The control law

$$u = -G\hat{x} = -\begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \tag{11.2-8}$$

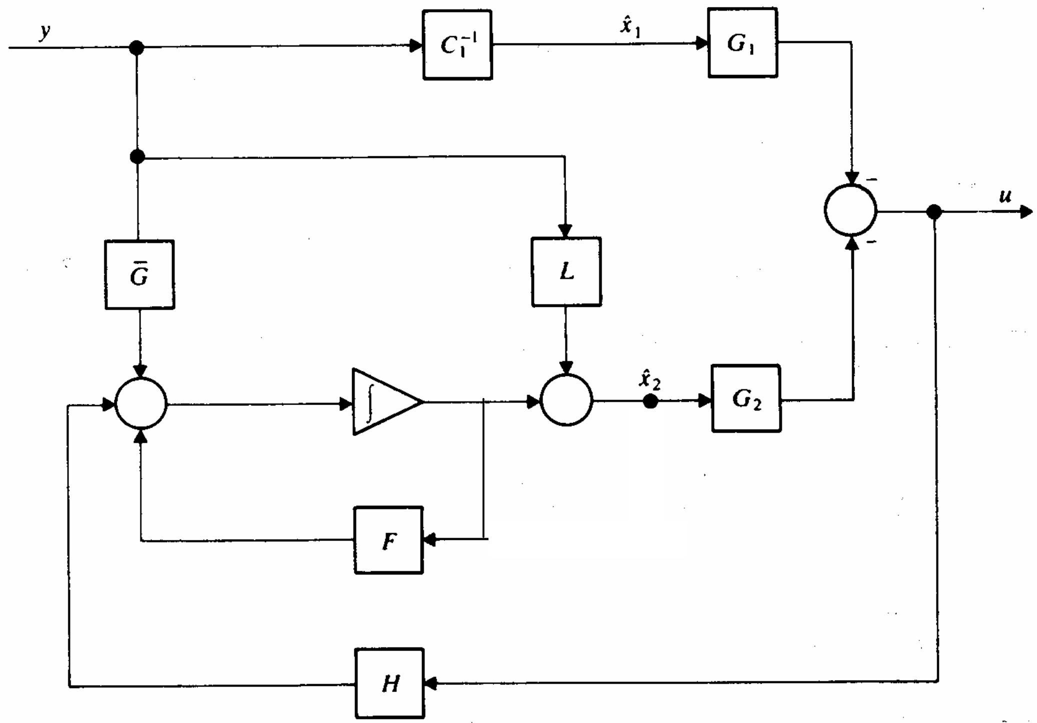


Figure 11.2-1 Block Diagram of Compensator Using Reduced-Order Observer

$$e = x - \hat{x} = \begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \end{bmatrix} \tag{11.2-9}$$

The dynamics of the plant

$$\dot{x} = Ax - BG(x - e) = (A - BG)x + B(G_1 e_1 + G_2 e_2) = (A - BG)x + BG_2 e_2 \tag{11.2-10}$$

$$\hat{x}_1 = x_1 \tag{11.2-11}$$

$$e_1 = 0 \tag{11.2-12}$$

$$\dot{e}_2 = Fe_2 \tag{11.2-13}$$

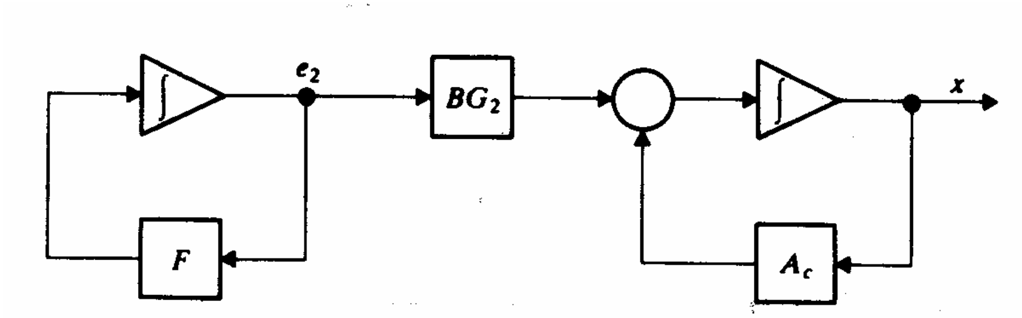


Figure 11.2-2 Block Diagram Representation of State and Error in System with Compensator Using Reduced-Order Observer

$$\begin{bmatrix} \dot{x} \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} A_c & BG \\ 0 & F \end{bmatrix} \begin{bmatrix} x \\ e_2 \end{bmatrix} \tag{11.2-14}$$

The combined plant characteristic equation

$$\begin{vmatrix} sI - A_c & -BG \\ 0 & sI - F \end{vmatrix} = \left|sI - A_c\right|\left|sI - F\right| = 0 \tag{11.2-15}$$

$$e_2(s) = (sI - F)^{-1} e_2(0) \tag{11.2-16}$$

$$x(s) = (sI - A_c)^{-1} BG_2 e_2(s) + (sI - A_c)^{-1} x(0) \tag{11.2-17}$$

$$x(s) = (sI - A_c)^{-1} BG_2 (sI - F)^{-1} e_2(0) + (sI - A_c)^{-1} x(0) \tag{11.2-18}$$

$$(sI - A_c)^{-1} BG_2 (sI - F)^{-1} = \frac{\mathrm{adj}(sI - A_c) BG_2 \mathrm{adj}(sI - F)}{|sI - A_c||sI - F|} \tag{11.2-19}$$