Optimal Control Theory

1 Linear Quadratic Regulator

Reasons of applying an optimal controller

- In a multiple-input or multiple-output system, there are **infinite solutions** by which the same closed-loop poles can be attained by using the pole-placement technique.
- The designer may not really know the desirable closed-loop pole locations. Choosing pole locations far from the origin may give very fast dynamic response but require control signals that are too large to be produced with the available power source. The control signal might be **saturated**. In such cases the closed-loop dynamic behavior will not be as predicted by the linear analysis, and may even be unstable.
- The process to be controlled may not be **controllable**. There may be some subspace of the process state-space in which the state vector cannot be moved around by application of suitable control signals. Hence design by pole placement will not work.

1.1 Formulation of the Optimal Control Problem

The dynamic process

$$\dot{x} = Ax + Bu \tag{1.1-1}$$

A linear control law

$$u = -Gx \tag{1.1-2}$$

The gain, G, that minimizes a specified performance criterion V or "cost function"

$$V = \int_{\tau}^{T} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt \tag{1.1-3}$$

Q and R: symmetric matrices

Q: the state weighting matrix

R: the control weighting matrix

- The quadratic term $x'Qx$ represents a penalty on the deviation of the state x from the origin.
- The quadratic term $u'Ru$ represents the cost of control.

Example: Suppose that x_1 represents the system error, and that x_2, …, x_k represent successive derivatives, i.e.,

$$\begin{aligned} x_2 &= \dot{x} \\ x_3 &= \ddot{x} \\ &\vdots \\ x_k &= x^{(k-1)} \end{aligned} \tag{1.1-4}$$

If only the error and none of its derivatives are of concern.

$$Q = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{1.1-5}$$

$$x'Qx = x_1^2 \tag{1.1-6}$$

To limit also the velocity, the performance integral might include a velocity penalty.

$$x'Qx = x_1^2 + c^2 x_2^2 \tag{1.1-7}$$

$$Q = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & c^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{1.1-8}$$

The system output

$$y = Cx \tag{1.1-9}$$

A system with a single output

$$y = c'x \tag{1.1-10}$$

$$y^2 = x'cc'x \tag{1.1-11}$$

$$Q = cc' \tag{1.1-12}$$

- The term $u'Ru$ in the performance index is included to limit the magnitude of the control signal u.

1.2 Quadratic Integrals and Matrix Differential Equations

The closed-loop dynamic behavior

$$\dot{x} = Ax - BGx = A_c x \tag{1.2-1}$$

$$A_c = A - BG \tag{1.2-2}$$

$$x(t) = \phi_c(t,\tau)x(\tau) \tag{1.2-3}$$

ϕ_c: the state-transition matrix corresponding to A_c

The performance index

$$V = \int_\tau^T [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt = \int_\tau^T x'(\tau)\phi_c'(t,\tau)\{Q + G'RG\}\phi_c(t,\tau)x(\tau)dt \tag{1.2-4}$$

$$V = x'(\tau)M(\tau,T)x(\tau) \tag{1.2-5}$$

$$M(\tau,T) = \int_\tau^T \phi_c'(t,\tau)\{Q + G'RG\}\phi_c(t,\tau)dt \tag{1.2-6}$$

M: a symmetric matrix

V is a function of the initial time τ.

$$V(\tau) = \int_\tau^T [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt = \int_\tau^T x'(t)L(t)x(t)dt \tag{1.2-7}$$

$$L = Q + G'RG \tag{1.2-8}$$

$$\frac{dV}{d\tau} = -x'(t)Lx(t)\,|_{t=\tau} = -x'(\tau)Lx(\tau) \tag{1.2-9}$$

But, from (1.2-5)

$$\frac{dV}{d\tau} = \dot{x}'(\tau)M(\tau,T)x(\tau) + x'(\tau)\dot{M}(\tau,T)x(\tau) + x'(\tau)M(\tau,T)\dot{x}(\tau) \tag{1.2-10}$$

$$\frac{dV}{d\tau} = x'(\tau)[A_c'(\tau)M(\tau,T) + \dot{M}(\tau,T) + M(\tau,T)A_c(\tau)]x(\tau) \tag{1.2-11}$$

$$-L = A_c'M + \dot{M} + MA_c \tag{1.2-12}$$

$$-\dot{M} = MA_c + A_c'M + L \tag{1.2-13}$$

- Equation (1.2-13) is an important differential equation. It appears in many forms in control theory and estimation.

$$M(\tau,T) = \int_{\tau}^{T} \phi_c'(t,\tau)L(\tau)\phi_c(t,\tau)dt \tag{1.2-14}$$

- Equation (1.2-13) is a first-order matrix differential equation and requires a boundary condition.

$$M(T,T) = 0 \tag{1.2-15}$$

1.3 The Optimum Gain Matrix

When any gain matrix G is chosen, the corresponding closed-loop performance

$$V = x'(\tau)M(\tau,T)x(\tau) \tag{1.3-1}$$

$$-\dot{M} = MA_c + A_c'M + L = M(A-BG) + (A'-G'B')M + Q + G'RG \tag{1.3-2}$$

- The task now is to find the matrix G which makes the solution to (1.3-2) as small as possible.

The optimum matrix $\hat{M}$ for any arbitrary initial state $x(\tau)$ and any matrix $M \neq \hat{M}$,

$$\hat{V} = x'\hat{M}x < x'Mx \tag{1.3-3}$$

$$-\dot{\hat{M}} = \hat{M}(A - B\hat{G}) + (A' - \hat{G}'B')\hat{M} + Q + \hat{G}'R\hat{G} \tag{1.3-4}$$

$$M = \hat{M} + N \tag{1.3-5}$$

$$G = \hat{G} + Z \tag{1.3-6}$$

$$-(\dot{\hat{M}} + \dot{N}) = (\hat{M} + N)[A - B(\hat{G} + Z)] + [A' - (\hat{G}' + Z')B'](\hat{M} + N) + Q + (\hat{G}' + Z')R(\hat{G} + Z) \tag{1.3-7}$$

$$-\dot{N} = NA_c + A_c'N + (\hat{G}'R - \hat{M}B)Z + Z'(R\hat{G} - B'\hat{M}) + Z'RZ \tag{1.3-8}$$

$A_c = A - BG = A - B(\hat{G} + Z)$

Compare with $-\dot{M} = MA_c + A_c'M + L$,

$$L = (\hat{G}'R - \hat{M}B)Z + Z'(R\hat{G} - B'\hat{M}) + Z'RZ \tag{1.3-9}$$

$$N(\tau,T) = \int_{\tau}^{T} \phi_c'(t,\tau) L(\tau) \phi_c(t,\tau) dt \tag{1.3-10}$$

$$x'Mx \le x'(\hat{M} + N)x = x'\hat{M}x + x'Nx \tag{1.3-11}$$

- The quadratic form $x'Nx$ must be positive definite, or at least positive semidefinite.
- $x'Nx$ is positive definite when the two linear terms in (1.3-9) absent.

$$R\hat{G} - B'\hat{M} = 0 \tag{1.3-12}$$

For nonsingular control weighting matrix R

$$\hat{G} = R^{-1}B'\hat{M} \tag{1.3-13}$$

$$-\dot{\hat{M}} = \hat{M}A + A'\hat{M} - \hat{M}BR^{-1}B'\hat{M} + Q \tag{1.3-14}$$

- A scalar first-order differential equation with a linear term and a quadratic term (as well as a constant term) is known as a *Riccati equation*.

With boundary condition,

$$\hat{M}(T,T) = 0 \tag{1.3-15}$$

1.4 The Steady State Solution

$$V_\infty = \int_\tau^\infty (x'Qx + u'Ru)dt \quad (1.4\text{-}1)$$

- In this case the terminal time, T, is infinite, the integration will either converge to a constant value or grow without limit.
- If it converges to a limit, the derivative $\dot{\hat{M}}$ tends to zero.

$$V_\infty = x'\overline{M}x \quad (1.4\text{-}2)$$

- $\overline{M}$ satisfies the algebraic quadratic equation (sometimes called the algebraic Riccati equation or ARE).

$$0 = \overline{M}A + A'\overline{M} - \overline{M}BR^{-1}B'\overline{M} + Q \quad (1.4\text{-}3)$$

The optimum gain in the steady state

$$\overline{G} = R^{-1}B'\overline{M} \quad (1.4\text{-}4)$$

For most design applications the following facts about the solution of (1.4-3) are sufficient.

(a) If the system is asymptotically stable, or

(b) If the system defined by the matrices (A, B) is controllable, and the system defined by (A, C) where $C'C = Q$, is observable,

Then the algebraic Riccati equation (ARE) has a *unique, positive definite* solution $\overline{M}$ which minimizes V_∞ when the control law $u = -R^{-1}B'\overline{M}x$ is used.

In the inverted pendulum problem, the state variables are $x_1 = \theta$ (angular position) and $x_2 = \dot{\theta}$ (angular velocity). The matrices defining the dynamics are

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \beta \end{bmatrix} \tag{1}$$

where

$$\alpha = -\frac{K_1 K_2}{JR},\ \beta = \frac{K_1}{JR},\ J = J_m + ml^2, \text{ and } \Omega^2 = \frac{mgl}{J_m + ml^2} = \frac{g}{l + J_m / ml} \tag{2}$$

A control law is sought to minimize the performance index

$$V = \int_{\tau}^{\infty} \left(\theta^2 + \frac{u^2}{c^2} \right) dt \tag{3}$$

The weighting matrices are seen to be

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, R = \left[\frac{1}{c^2} \right] \tag{4}$$

Let the performance matrix $\hat{M}$ be given by

$$\hat{M} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \tag{5}$$

The gain matrix is

$$\hat{G} = R^{-1} B' \hat{M} = \left[c^2 \right] \begin{bmatrix} 0 & \beta \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = \begin{bmatrix} c^2 \beta m_2 & c^2 \beta m_3 \end{bmatrix} \tag{6}$$

The terms needed for the matrix quadratic are

$$\hat{M}A = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix} = \begin{bmatrix} m_2\Omega^2 & m_1 - m_2\alpha \\ m_3\Omega^2 & m_2 - m_3\alpha \end{bmatrix} \tag{7}$$

$$A'\hat{M} = (\hat{M}A)' = \begin{bmatrix} m_2\Omega^2 & m_3\Omega^2 \\ m_1 - m_2\alpha & m_2 - m_3\alpha \end{bmatrix} \tag{8}$$

$$\hat{M}BR^{-1}B'\hat{M} = \hat{M}B\hat{G} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \begin{bmatrix} 0 \\ \beta \end{bmatrix} \begin{bmatrix} c^2\beta m_2 & c^2\beta m_3 \end{bmatrix} = \begin{bmatrix} c^2\beta^2 m_2^2 & c^2\beta^2 m_2 m_3 \\ c^2\beta^2 m_2 m_3 & c^2\beta^2 m_3^2 \end{bmatrix} \tag{9}$$

By substitution into

$$0 = \overline{M}A + A'\overline{M} - \overline{M}BR^{-1}B'\overline{M} + Q \tag{10}$$

Thus, the individual terms are

$$0 = 2m_2\Omega^2 - c^2\beta^2 m_2^2 + 1 \tag{11}$$

$$0 = m_1 - m_2\alpha + m_3\Omega^2 - c^2\beta^2 m_2 m_3 \tag{12}$$

$$0 = 2m_2 - 2m_3\alpha - c^2\beta^2 m_3^2 \tag{13}$$

Solving for m_1, m_2, and m_3,

$$m_2 = \frac{\Omega^2 \pm \sqrt{\Omega^4 + c^2\beta^2}}{c^2\beta^2} \tag{14}$$

$$m_3 = \frac{-\alpha \pm \sqrt{\alpha^2 + 2(\Omega^2 \pm \sqrt{\Omega^4 + c^2\beta^2})}}{c^2\beta^2} \tag{15}$$

$$m_1 = \frac{\alpha\Omega^2 + \sqrt{(\Omega^4 + c^2\beta^2)(\alpha^2 + 2(\Omega^2 \pm \sqrt{\Omega^4 + c^2\beta^2}))}}{c^2\beta^2} \tag{16}$$

Solving for the gains g_1 and g_2,

$$g_1 = \frac{\Omega^2 \pm \sqrt{\Omega^4 + c^2\beta^2}}{\beta} \tag{17}$$

$$g_2 = \frac{-\alpha \pm \sqrt{\alpha^2 + 2(\Omega^2 \pm \sqrt{\Omega^4 + c^2\beta^2})}}{\beta} \tag{18}$$

The correct signs in (14), (15), (16), (17) and (18) are the ones which make real solutions.

If α is very small and negligible and β is unity. The only solutions will be

$$m_2 = \frac{\Omega^2 + \sqrt{\Omega^4 + c^2}}{c^2},\ m_3 = \frac{\sqrt{2(\Omega^2 + \sqrt{\Omega^4 + c^2})}}{c^2},\ m_1 = \frac{\sqrt{(\Omega^4 + c^2)(2(\Omega^2 + \sqrt{\Omega^4 + c^2}))}}{c^2},$$

$$g_1 = \Omega^2 + \sqrt{\Omega^4 + c^2}, \text{ and } g_2 = \sqrt{2(\Omega^2 + \sqrt{\Omega^4 + c^2})} \tag{19}$$

The closed-loop matrix is

$$A_c = A - BG = \begin{bmatrix} 0 & 1 \\ -\sqrt{\Omega^4 + c^2} & -\sqrt{2(\Omega^2 + \sqrt{\Omega^4 + c^2})} \end{bmatrix} \tag{20}$$

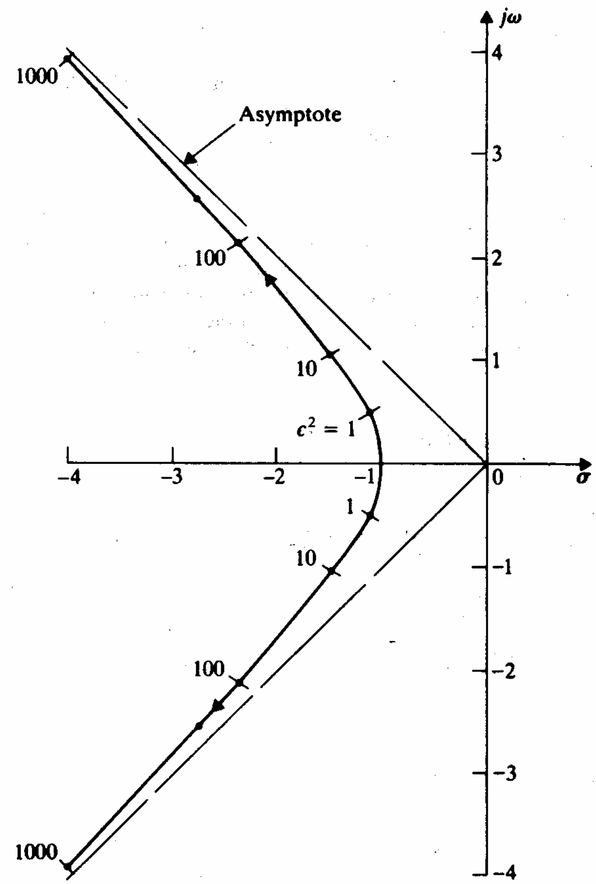
And the characteristic equation is

$$s^2 + \sqrt{2(\Omega^2 + \sqrt{\Omega^4 + c^2})}s + \sqrt{\Omega^4 + c^2} = 0 \tag{21}$$

The roots of which are

$$s_1, s_2 = \sqrt{\frac{\overline{\Omega}^2 + \Omega^2}{2}} \pm \sqrt{\frac{\overline{\Omega}^2 - \Omega^2}{2}} \quad \text{where } \overline{\Omega}^2 = \sqrt{\Omega^4 + c^2} \tag{22}$$

The locus of closed-loop poles as the weighting factor c is varied from ∞ to 0 is shown in the figure below.



The following characteristics of the locus are noteworthy, $V = \int_{\tau}^{\infty} \left(\theta^2 + \frac{u^2}{c^2} \right) dt$.

(a) As c increases, the closed-loop roots tend to asymptotes at 45° to the real axis, and move out to ∞ along these asymptotes. This implies that the response time tends to zero and the damping factor tends to $\zeta = 1/2^{1/2} = 0.707$.

(b) As c tends to zero, the cost of control tends to become very large. If the open-loop system were stable, and it would turn out that the gains g_1 and g_2 would tend to zero and the open-loop system would "coast" to rest, without incurring any control cost. In the present case, however, the open-loop system is unstable, and cannot coast to rest without control. A certain amount of control is necessary to stabilize the system. The second closed-loop pole also tends to $s = -\Omega$ is a consequence of a general result that as the control weighting becomes very large, the closed loop poles corresponding to unstable open loop poles tend to their mirror images with respect to the imaginary axis. In other words, if $s_i = +\alpha + j\beta$ ($\alpha \geq 0$) in the open-loop system, then the corresponding pole in the closed-loop system tends to $\bar{s}_i = -\alpha + j\beta$. This is a general property of optimal control laws.

1.5 Disturbances and Reference Inputs: Exogenous Variables

$$\dot{x} = Ax + Bu + Ex_0 \tag{1.5-1}$$

$$\dot{x}_0 = A_0 x_0 \tag{1.5-2}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{1.5-3}$$

$$\mathbf{x} = \begin{bmatrix} e \\ - \\ x_0 \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} A & | & E \\ - & - & - \\ 0 & | & A_0 \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} B \\ - \\ 0 \end{bmatrix} \tag{1.5-4}$$

Obviously, the exogenous state x_0 is not controllable; hence an appropriate performance integral would be

$$V = \int_{\tau}^{T} (x'Qx + u'Ru)dt \tag{1.5-5}$$

$$\mathbf{Q} = \begin{bmatrix} Q & | & 0 \\ - & - & - \\ 0 & | & 0 \end{bmatrix} \tag{1.5-6}$$

- The upper limit on the integral is intentionally not made infinite. It may not be possible to achieve a steady state error of zero with a control u that also goes to zero. The cost function will become infinite as $T \to \infty$.
- One way of approaching this problem is to find a control $\bar{u}$ which satisfies the requirements of zero steady state error.

For $x = \dot{x} = 0$, the required steady state control $\bar{u}$ must satisfy

$$B\bar{u} + Ex_0 = 0 \tag{1.5-7}$$

- The total control u is the sum of the steady state control and a "corrective" control v.

$$u = \bar{u} + v \tag{1.5-8}$$

$$\dot{x} = Ax + Bu + Ex_0 = Ax + Bv \tag{1.5-9}$$

$$\bar{V} = \int_{\tau}^{\infty} (x'Qx + v'Rv)dt \tag{1.5-10}$$

The performance matrix $\hat{\mathbf{M}}$ for the metasystem

$$\hat{\mathbf{M}} = \begin{bmatrix} \hat{M}_1 & | & \hat{M}_2 \\ - & - & - \\ \hat{M}_2' & | & \hat{M}_3 \end{bmatrix} \tag{1.5-11}$$

The gain matrix $\hat{\mathbf{G}}$ for the metasystem

$$\hat{\mathbf{G}} = R^{-1}\begin{bmatrix} B' & | & 0 \end{bmatrix} \begin{bmatrix} \hat{M}_1 & | & \hat{M}_2 \\ - & - & - \\ \hat{M}_2' & | & \hat{M}_3 \end{bmatrix} = \begin{bmatrix} R^{-1}B'\hat{M}_1 & | & R^{-1}B'\hat{M}_2 \end{bmatrix} \tag{1.5-12}$$

From $-\dot{\hat{\mathbf{M}}} = \hat{\mathbf{M}}\mathbf{A} + \mathbf{A}'\hat{\mathbf{M}} - \hat{\mathbf{M}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\hat{\mathbf{M}} + \mathbf{Q}$,

$$-\dot{\hat{M}}_1 = \hat{M}_1 A + A'\hat{M}_1 - \hat{M}_1 BR^{-1}B'\hat{M}_1 + Q \tag{1.5-13}$$

$$-\dot{\hat{M}}_2 = \hat{M}_1 E + \hat{M}_2 A_0 + (A' - \hat{M}_1 BR^{-1}B')\hat{M}_2 \tag{1.5-14}$$

$$-\dot{\hat{M}}_3 = \hat{M}_3 A_0 + A_0'\hat{M}_3 + \hat{M}_2' E + E'\hat{M}_2 - \hat{M}_2' BR^{-1}B'\hat{M}_2 \tag{1.5-15}$$

Owing to the special structure of **A**, **B**, and **Q**, the following facts about the submatrices of $\hat{\mathbf{M}}$ emerge:

(a) The solution for $\hat{M}_1$, and hence the corresponding gain $R^{-1}B'\hat{M}_1$, is the same as it would have been with x_0 absent from the problem.

(b) The differential equation for $\hat{M}_2$, from which the gain $R^{-1}B'\hat{M}_2$ is determined, does not depend on $\hat{M}_3$, and in fact is a linear equation.

$$-\dot{\hat{M}}_2 = \hat{M}_1 E + \hat{M}_2 A_0 + A_c' \hat{M}_2 \quad (1.5\text{-}16)$$

$$A_c = A - BR^{-1}B'\hat{M}_1 \quad (1.5\text{-}17)$$

A steady state solution

$$0 = \overline{M}_1 E + \overline{M}_2 A_0 + A_c' \overline{M}_2 \quad (1.5\text{-}18)$$

The necessary gains to realize the control law

$$u = -R^{-1}B'\overline{M}_1 x - R^{-1}B'\overline{M}_2 x_0 \quad (1.5\text{-}19)$$

(c) The differential equation for $\hat{M}_3$ is also linear. Whether it has a steady state solution depends on A_0. If $A_0 = 0$, (1.5-15) does not have a steady state solution. But this doesn't matter because $\hat{M}_3$ is not used in the determination of the gain matrix.

For $A_0 = 0$,

$$-\dot{\hat{M}}_2 = \hat{M}_1 E + A_c' \hat{M}_2 \tag{1.5-20}$$

$$-\dot{\hat{M}}_3 = \hat{M}_2' E + E' \hat{M}_2 - \hat{M}_2' B R^{-1} B' \hat{M}_2 \tag{1.5-21}$$

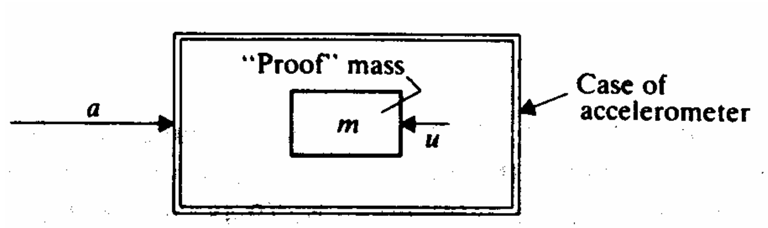
$$M_3(t) = M_3(T) + \int_{\tau}^{T} (\hat{M}_2' E + E' \hat{M}_2 - \hat{M}_2' B R^{-1} B' \hat{M}_2) dt \tag{1.5-22}$$

Steady state solution,

$$\overline{M}_2 = -(A_c')^{-1} \overline{M}_1 E \tag{1.5-23}$$

$$G_0 = -R^{-1} B' \overline{M}_2 = -R^{-1} B' (A_c')^{-1} \overline{M}_1 E = B^* E \tag{1.5-24}$$

$$B^* = -R^{-1} B' (A_c')^{-1} \overline{M}_1 \tag{1.5-25}$$



The differential equations governing the displacement of the proof mass in an accelerometer, shown above, is given by

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = -\frac{K}{m}x_1 - \frac{B}{m}x_2 + a \tag{2}$$

Suppose that the spring and damping forces are both absent. To keep the proof mass from striking the wall, capture force, u, is used to capture the proof mass. The differential equations for the proof mass, with the acceleration due to the capture force are

$$\dot{x}_1 = x_2 \tag{3}$$

$$\dot{x}_2 = u + a \tag{4}$$

First, consider the control problem of returning the proof mass to the origin ($x_1 = x_2 = 0$) in the absence of an input acceleration ($a = 0$). The matrices for the dynamics are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5}$$

We use a performance criterion of the form

$$V = \int_{\tau}^{\infty} (x_1^2 + \frac{u^2}{c^2}) dt \tag{6}$$

The gain matrix for this control design is

$$G = R^{-1} B' M = [c^2][0 \quad 1] \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = [c^2 m_2 \quad c^2 m_3] \tag{7}$$

and the components of M is determined from $0 = \overline{M}A + A'\overline{M} - \overline{M}BR^{-1}B'\overline{M} + Q$.

$$0 = -c^2 m_2^2 + 1 \tag{8}$$

$$0 = m_1 - c^2 m_2 m_3 \tag{9}$$

$$0 = 2m_2 - c^2 m_3^2 \tag{10}$$

The solutions are

$$m_1 = \sqrt{\frac{2}{c}}, \; m_2 = \frac{1}{c}, \text{ and } m_3 = \sqrt{\frac{2}{c^3}} \tag{11}$$

Thus, the gain matrix becomes

$$G = [c^2 m_2 \quad c^2 m_3] = [c \quad \sqrt{2c}] \tag{12}$$

The dynamic matrix of the closed-loop system is given by

$$A_c = A - BG = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [c \quad \sqrt{2c}] = \begin{bmatrix} 0 & 1 \\ -c & -\sqrt{2c} \end{bmatrix} \tag{13}$$

Hence the closed-loop poles are the roots of

$$|sI - A_c| = \begin{vmatrix} s & -1 \\ c & s + \sqrt{2c} \end{vmatrix} = s^2 + \sqrt{2c}\,s + c = 0 \tag{14}$$

or

$$s_{1,2} = -\frac{\sqrt{2c}}{2} \pm \frac{\sqrt{2c}}{2} j \tag{15}$$

The locus of the closed-loop poles are thus straight lines at 45 degrees to the coordinate axes and moving away from the origin as $c \to \infty$.

The case we really want to consider, of course, is a nonzero external acceleration. Any model for a can be used (e.g., a step, a ramp, etc.). Suppose that it is modeled as a step

$$\dot{a} = 0 \tag{16}$$

Adjoining this to (3) and (4) gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ - \\ \dot{a} \end{bmatrix} = \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \\ - & - & - & - \\ 0 & 0 & | & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ - \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ - \\ 0 \end{bmatrix} u \tag{17}$$

In particular, let

$$\hat{\mathbf{M}} = \begin{bmatrix} \hat{M}_1 & | & \hat{M}_2 \\ - & - & - \\ \hat{M}_2' & | & \hat{M}_3 \end{bmatrix} = \begin{bmatrix} m_1 & m_2 & | & m_4 \\ m_2 & m_3 & | & m_5 \\ - & - & - & - \\ m_4 & m_5 & | & m_6 \end{bmatrix} \tag{18}$$

Then, as already found,

$$\hat{M}_1 = \hat{M} = \begin{bmatrix} \sqrt{\frac{2}{c}} & \frac{1}{c} \\ \frac{1}{c} & \sqrt{\frac{2}{c^3}} \end{bmatrix} \tag{19}$$

The submatrix $\overline{M}_2$ is found using (1.5-23). ln this application (1.5-20) is

$$\overline{M}_2 = -(A_c')^{-1}\overline{M}_1 E = \begin{bmatrix} m_4 \\ m_5 \end{bmatrix} = -\begin{bmatrix} 0 & -c \\ 1 & -\sqrt{2c} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\frac{2}{c}} & \frac{1}{c} \\ \frac{1}{c} & \sqrt{\frac{2}{c^3}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{c^2} \end{bmatrix} \tag{20}$$

The gain due to the forcing acceleration is

$$G_a = R^{-1}B'\overline{M}_2 = [c^2][0 \quad 1]\begin{bmatrix} 0 \\ \frac{1}{c^2} \end{bmatrix} = 1 \tag{21}$$

Thus the control law

$$u = -g_1 x_1 - g_2 x_2 - G_a a = -cx_1 - \sqrt{2c}\, x_2 - a \tag{22}$$

Note that we never needed to determine the remaining term m_6 of $\overline{\mathbf{M}}$. The differential equation for m_6 is determined from $-\dot{\hat{M}}_3 = \hat{M}_2'E + E'\hat{M}_2 - \hat{M}_2'BR^{-1}B'\hat{M}_2$.

$$-\dot{m}_6 = 2m_5 - c^2 m_5^2 = \frac{1}{c^2} \tag{23}$$

$$m_6(\tau) = m_6(T) + \frac{T-\tau}{c^2} \tag{24}$$

A steady state solution for m_6 does not exist. This is not surprising, in view of the fact that a constant value of external acceleration demands a constant, nonzero control, and this cannot result in a finite value of the performance integral V over an infinite time interval.

1.6 General Performance Integral

When the performance integral includes also a cross term, $2x'S'u = x'S'u + u'Sx$, the optimum gain

$$\hat{G} = R^{-1}(B'\hat{M} + S) \tag{1.6-1}$$

$$-\dot{\hat{M}} = \hat{M}\overline{A} + \overline{A}'\hat{M} - \hat{M}BR^{-1}B'\hat{M} + \overline{Q} \tag{1.6-2}$$

$$\overline{A} = A - BR^{-1}S \tag{1.6-3}$$

$$\overline{Q} = Q - S'R^{-1}S \tag{1.6-4}$$

Proof:

$$u = v - R^{-1}Sx \tag{1.6-5}$$

$$\dot{x} = Ax + Bu = (A - BR^{-1}S)x + Bv = \overline{A}x + Bv \tag{1.6-6}$$

$$V = \int_{\tau}^{T} (x'Qx + x'S'u + u'Sx + u'Ru)dt \tag{1.6-7}$$

$$x'Qx + x'S'(v - R^{-1}Sx) + (v' - x'S'R^{-1})Sx + (v' - x'S'R^{-1})R(v - R^{-1}Sx) = x'(Q - S'RS)x + v'Rv \tag{1.6-8}$$

$$V = \int_{\tau}^{T} (x'\overline{Q}x + v'Rv)dt \tag{1.6-9}$$

$$v = -\overline{G}x \tag{1.6-10}$$

$$\overline{G} = R^{-1}B'\hat{M} \tag{1.6-11}$$

$$u = -(R^{-1}B'\hat{M} + R^{-1}S)x = -\hat{G}x \tag{1.6-13}$$

1.7 Weighting of Performance at Terminal Time

- In control processes of finite time duration, the terminal state $x(T)$ is often as important as, or more important than, the manner in which the state is reached.

$$V = \int_{\tau}^{T} [x'(t)Qx(t) + u'(t)Ru(t)]dt + x'(T)Zx(T) \tag{1.7-1}$$

$x'(T)Zx(T)$: a terminal penalty, the cost of not getting to the origin at the terminal time

$$\hat{M}(T,T) = Z \tag{1.7-2}$$

$$x(T) = \phi_c(T,\tau)x(\tau) \tag{1.7-3}$$

$$x'(T)Zx(T) = x'(\tau)\phi_c'(T,\tau)Z\phi_c(T,\tau)x(\tau) \tag{1.7-4}$$

$$V = V(\tau,T) = x'(\tau)M(\tau,T)x(\tau) \tag{1.7-5}$$

$$M(\tau,T) = \int_{\tau}^{T} [x'(t)Qx(t) + u'(t)Ru(t)]dt + \phi_c'(T,\tau)Z\phi_c(T,\tau) \tag{1.7-6}$$

$$M(T,T) = Z \tag{1.7-7}$$

since $\phi_c(T,T) = I$ for any transition matrix,

$$\frac{\partial V}{\partial \tau} = \dot{x}'(\tau)M(\tau,T)x(\tau) + x'(\tau)\dot{M}(\tau,T)x(\tau) + x'(\tau)M(\tau,T)\dot{x}(\tau) \tag{1.7-8}$$

From $V = \int_{\tau}^{T} [x'(t)Qx(t) + u'(t)Ru(t)]dt + x'(T)Zx(T)$ and $u = -Gx$,

$$\frac{\partial V}{\partial \tau} = x'(\tau)Lx(\tau) + \frac{\partial}{\partial \tau}[x'(\tau)\phi_c'(T,\tau)Z\phi_c(T,\tau)x(\tau)] \tag{1.7-9}$$

where $L = Q + G'RG$.

From $x(T) = \phi_c(T,\tau)x(\tau)$,

$$\frac{\partial x(T)}{\partial \tau} = \frac{\partial \phi_c(T,\tau)}{\partial \tau}x(\tau) + \phi_c(T,\tau)\dot{x}(\tau) \tag{1.7-10}$$

For any transition matrix

$$\phi_c(\tau,T)\phi_c(T,\tau) = I \tag{1.7-11}$$

$$\frac{\partial \phi_c(\tau,T)}{\partial \tau}\phi_c(T,\tau) + \phi_c(\tau,T)\frac{\partial \phi_c(T,\tau)}{\partial \tau} = 0 \tag{1.7-12}$$

From $\dot{x} = A_c x$,

$$\frac{\partial \phi_c(\tau,T)}{\partial \tau} = A_c(\tau)\phi_c(\tau,T) \tag{1.7-13}$$

$$\frac{\partial \phi_c(\tau,T)}{\partial \tau}\phi_c(T,\tau) + \phi_c(\tau,T)\frac{\partial \phi_c(T,\tau)}{\partial \tau} = A_c(\tau) + \phi_c(\tau,T)\frac{\partial \phi_c(T,\tau)}{\partial \tau} = 0 \tag{1.7-14}$$

$$\frac{\partial \phi_c(T,\tau)}{\partial \tau} = -\phi_c(T,\tau)A_c(\tau) \tag{1.7-15}$$

$$\dot{x}(\tau) = A_c(\tau)x(\tau) \tag{1.7-16}$$

$$\frac{\partial x(T)}{\partial \tau} = \frac{\partial \phi_c(T,\tau)}{\partial \tau} x(\tau) + \phi_c(T,\tau)\dot{x}(\tau) = 0 \tag{1.7-17}$$

- Hence the second term in $\frac{\partial V}{\partial \tau} = x'(\tau)Lx(\tau) + \frac{\partial}{\partial \tau}[x'(\tau)\phi_c'(T,\tau)Z\phi_c(T,\tau)x(\tau)]$ vanishes and $M(t, T)$ satisfies the same differential equation as before, namely

$$-\dot{M} = MA_c + A_c'M + L \tag{1.7-18}$$

but subject to the condition $M(T,T) = Z$.

The approximate dynamic model of a missile which is controlled by the use of a control acceleration normal to the velocity vector is represented by

$$\dot{z} = (T - \tau)u \tag{1}$$

where z is the projected miss distance between the missile and the target, u is the normal acceleration, and T - τ is the time-to-go, assumed to be a known quantity.

If z is brought to zero at any time, the missile will, in the absence of any further normal acceleration (u = 0), continue on a straight-line trajectory to intercept the target. Thus the control objective is to reduce z to zero. There are of course countlessly many ways that this can be accomplished. The only requirement is that

$$z(T) = z(\tau) + \int_{\tau}^{T} (T - t)u(t)dt = 0 \tag{2}$$

In order to formulate a suitable optimization problem we suppose that the control objective is to minimize

$$V = \int_{\tau}^{T} u^2(t)dt + k^2 z^2(T) \tag{3}$$

The integral term in (3) is a quadratic form in the normal acceleration; it penalizes large accelerations and hence is a way of limiting the acceleration requirement. The second term penalizes the terminal miss distance. The larger the value of k the greater the cost attached to missing the target; as $k \to \infty$ the target must be hit at all costs.

The matrices that define the problem are all scalars

$$A = 0, B(\tau) = T - \tau, Q = 0, R = I, \text{and } Z = k^2 \tag{4}$$

Thus the "optimum" guidance law is

$$u(\tau) = -Gz(\tau) = -R^{-1}B(\tau)M(\tau,T)z(\tau) = -(T-\tau)M(\tau,T)z(\tau) \quad (5)$$

where $M(\tau, T)$ is a scalar satisfying the Riccati equation

$$-\dot{M} = -(T-\tau)^2 M^2 \quad (6)$$

subject to the terminal condition

$$M(T,T) = k^2 \quad (7)$$

To solve (7) let

$$W(\tau) = 1/M(\tau,T) \quad (8)$$

Then

$$\dot{W} = (T-\tau)^2 \quad (9)$$

which is integrated directly to give

$$W(T) = W(\tau) + \int_{\tau}^{T}(T-t)^2 dt = W(\tau) + \frac{(T-\tau)^3}{3} \quad (10)$$

But, by (7) and (8), $W(T) = 1/k^2$. Thus

$$W(\tau) = \frac{1}{k^2} - \frac{(T-\tau)^3}{3} \quad (11)$$

The desired solution to the Riccati equation is

$$M(\tau,T) = \frac{3}{3/k^2 - (T-\tau)^3} \tag{12}$$

If we truly want the terminal miss to be zero, we must let k^2 be infinite, in this limiting case (12) becomes

$$M(\tau,T) = \frac{3}{-(T-\tau)^3} \tag{13}$$

The guidance law in (5) becomes $u(\tau) = \frac{3}{(T-\tau)^2} z(\tau)$.

2 Random Processes

- The disturbances to a real system in many cases are random processes.
- The sensors used in the measurement of the system output, y, in many cases are not perfect, and subject to errors, which are also random processes.
- Since the disturbances and sensor errors are random processes, the response of the system, either open-loop, or with the feedback control present, is also a random process.

$$\dot{x} = Ax + Bu + Fv \tag{2-1}$$

$$y = Cx + w \tag{2-2}$$

v and w: random processes.

2.1 Conceptual Models for Random Processes

Let $x_i(t)$ denote the ith member of an ensemble of N members.

Mean: $$\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t) \tag{2.1-1}$$

Mean Square: $$\overline{x^2(t)} = \frac{1}{N}\sum_{i=1}^{N} x_i^2(t) \tag{2.1-2}$$

Variance: $$v(t) = \frac{1}{N}\sum_{i=1}^{N} [x_i(t) - \bar{x}(t)]^2 \tag{2.1-3}$$

Correlation Function: $$r(t,\tau) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)x_i(\tau) \tag{2.1-4}$$

2.2 Statistical Characteristics of Random Processes

First- and Second- Order Statistics: In theory, a random process is characterized by an infinite series of joint probability density functions

$$pdf[x;t]$$
$$pdf[x_1, x_2; t_1, t_2]$$
$$pdf[x_1, x_2, x_3; t_1, t_2, t_3]$$
$$\vdots \qquad (2.2\text{-}1)$$

- Each density function describes the probability of finding x somewhere at some time.

Example: $pdf[x_1, x_2, x_3; t_1, t_2, t_3]\Delta x_1 \Delta x_2 \Delta x_3 = prob[x_1 < x(t_1) < x_1 + \Delta x_1, x_2 < x(t_2) < x_2 + \Delta x_2, x_3 < x(t_3) < x_3 + \Delta x_3]$

Mean: $$\mu(t) = E\{x(t)\} = \int_{-\infty}^{\infty} (x) pdf[x,t]dx \qquad (2.2\text{-}2)$$

Mean Square: $$E\{x^2(t)\} = \int_{-\infty}^{\infty} (x^2) pdf[x,t]dx \qquad (2.2\text{-}3)$$

Variance: $$\sigma^2(t) = E\{[x(t) - \mu(t)]^2\} = \int_{-\infty}^{\infty} [x - \mu(t)]^2 pdf[x,t]dx \qquad (2.2\text{-}4)$$

Correlation Function: $$\rho(t,\tau) = E\{x(t)x(\tau)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 x_2) pdf[x_1, x_2; t, \tau]dx_1 dx_2 \qquad (2.2\text{-}5)$$

$E\{\ \}$: mathematical expectation

When $\tau = t$ the correlation function

$$\rho(t,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1 x_2)pdf\,[x_1, x_2; t, t]dx_1 dx_2 = \int_{-\infty}^{\infty}(x_1^2)pdf\,[x_1; t]dx_1 = E\{x^2(t)\} \qquad (2.2\text{-}6)$$

For vector processes,

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad (2.2\text{-}7)$$

$$\mu(t) = E\{x(t)\} = \begin{bmatrix} E\{x_1(t)\} \\ \vdots \\ E\{x_n(t)\} \end{bmatrix} \qquad (2.2\text{-}8)$$

$$R(t,\tau) = E\{x(t)x'(\tau)\} = \begin{bmatrix} E\{x_1(t)x_1(\tau)\} & \cdots & E\{x_1(t)x_n(\tau)\} \\ \vdots & \ddots & \vdots \\ E\{x_n(t)x_1(\tau)\} & \cdots & E\{x_n(t)x_n(\tau)\} \end{bmatrix} \qquad (2.2\text{-}9)$$

- The diagonal entries in the correlation matrix are the autocorrelation functions.
- The off-diagonal terms in the correlation matrix are cross-correlation.

$$R(t,\tau) = R'(\tau,t) \qquad (2.2\text{-}10)$$

$$R(t,t) = E\{x(t)x'(t)\} = R'(t,t) = P(t) \qquad (2.2\text{-}11)$$

$R(t, t)$, $P(t)$: the covariance matrix for the vector process $x(t)$

- If the covariance matrix, $P(t)$, is diagonal, the components of the vector x are said to be uncorrelated.

$$P(t) = R(t,t) = \begin{bmatrix} r_{11}(t,t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_{nn}(t,t) \end{bmatrix} \quad (2.2\text{-}12)$$

$$E\{x_i(t)x_j(t)\} = 0 \text{ for } i \neq j \quad (2.2\text{-}13)$$

Stationary and Ergodic Processes: The set of probability density functions are general functions of the time variables t_1, t_2, t_3, ... for a general random process. If the functions are invariant to a translation of time, the process is called **stationary**.

$$pdf[x;t+\tau] = pdf[x;t] \text{ for all } \tau$$

$$pdf[x_1,x_2;t_1+\tau,t_2+\tau] = pdf[x_1,x_2;t_1,t_2] \quad \text{for all } \tau$$

$$\vdots \quad (2.2\text{-}14)$$

If a process is **ergodic**, it is stationary and a single sample function is representative of the ensemble.

For ergodic processes,

Mean: $$\mu(t) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)dt \quad (2.2\text{-}15)$$

Variance: $$\sigma^2(t) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} [x-\mu(t)]^2 dt \quad (2.2\text{-}16)$$

Correlation Function: $$\rho(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt \quad (2.2\text{-}17)$$

2.3 Power Spectral Density Function

- One of the most useful descriptions of a random process is its **power spectral density function**, $S(\omega)$, also called the **power spectrum**, which is defined as the Fourier transform of the correlation function.

$$S(\omega) = \int_{-\infty}^{\infty} \rho(\tau) e^{-j\omega\tau} d\tau \tag{2.3-1}$$

Methods to determine power spectral density function

1. Computing a correlation function by multiplying $x(t)$ by $x(t+\tau)$ and integration.
2. Measuring $S(\omega)$ by connecting the output of the process to a device known as a spectrum analyzer and then measuring the power contained in the random signal in different frequency bands. A spectrum analyzer is actually a sharply tuned filter with an adjustable center frequency.

Consider a function of frequency approximated by

$$|X_T(j\omega)|^2 = \left| \int_{-T/2}^{T/2} x(t) e^{-j\omega t} dt \right|^2 \tag{2.3-2}$$

- The spectrum analyzer produces an approximation to the Fourier transform of the signal.

$$|X_T(j\omega)|^2 = X_T(j\omega) X_T(-j\omega) = \int_{-T/2}^{T/2} x(t) e^{-j\omega t} dt \cdot \int_{-T/2}^{T/2} x(\tau) e^{j\omega\tau} d\tau = \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} x(t) x(\tau) e^{-j\omega(t-\tau)} dt d\tau \tag{2.3-3}$$

Define $\lambda = t\text{-}\tau$,

$$\frac{1}{T}|X_T(j\omega)|^2 = \int_{-\tau-T/2}^{-\tau+T/2}\left\{\frac{1}{T}\int_{-T/2}^{T/2} x(\tau+\lambda)x(\tau)d\tau\right\}e^{-j\omega\lambda}d\lambda \quad (2.3\text{-}4)$$

If the process is ergodic,

$$\rho_T(\lambda) = \frac{1}{T}\int_{-T/2}^{T/2} x(\tau+\lambda)x(\tau)d\tau \to \rho(\lambda) \text{ as } T\to\infty \quad (2.3\text{-}5)$$

$$\lim_{T\to\infty}\frac{1}{T}|X_T(j\omega)|^2 = \lim_{T\to\infty}\int_{-\tau-T/2}^{-\tau+T/2}\rho_T(\lambda)e^{-j\omega\lambda}d\lambda = S(\omega) \quad (2.3\text{-}6)$$

- The power spectral density function varies with the limit of the magnitude square of the ordinary Fourier transform of the signal.
- The Fourier transform of a signal describes how its energy is distributed in frequency; division by T converts energy to power.
- The correlation function is the inverse Fourier transform of the power spectral density.

$$\rho(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}d\omega \quad (2.3\text{-}7)$$

For $\tau = 0$

$$\rho(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega \quad (2.3\text{-}8)$$

$$\rho(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt \qquad (2.3\text{-}9)$$

- The area under the spectral density function is 2π times the mean square value of the random process.
- The spectral density can be expressed as a function of frequency, $f = \omega/(2\pi)$.

$$\rho(0) = \int_{-\infty}^{\infty} \bar{S}(f)df = 2\int_{0}^{\infty} \bar{S}(f)df \qquad (2.3\text{-}10)$$

$\bar{S}(f)$: the spectral density in (units)^2/Hz.

2.4 White Noise and Linear System Response

- White noise is a random process with an expected value (mean) of zero with an absolutely flat power spectrum.

$$S(\omega) = W = \text{constant for all } \omega \tag{2.4-1}$$

The correlation function of white noise

$$\rho(\tau) = W\delta(\tau) \tag{2.4-2}$$

$\delta(\tau)$: a unit impulse

- A vector random process is white noise if its correlation matrix is of the from

$$R(\tau) = W\delta(\tau) = E\{x(t)x'(t+\tau)\} \tag{2.4-3}$$

W: a square matrix.

Consider a linear system, the input to which is the signal $u(t)$ and the output from which is $y(t)$.

$$y(t) = \int_0^t H(t,\lambda)u(\lambda)d\lambda \tag{2.4-4}$$

The correlation matrix for the output $y(t)$

$$R_y(t,\tau) = E\{y(t), y'(\tau)\} = E\left\{\int_0^t H(t,\lambda)u(\lambda)d\lambda \cdot \int_0^\tau u'(\xi)H'(\tau,\xi)d\xi\right\}$$

$$= E\left\{\int_0^t\int_0^\tau H(t,\lambda)u(\lambda)u'(\xi)H'(\tau,\xi)d\lambda d\xi\right\} \tag{2.4-5}$$

For white noise input, $u(t)$,

$$E\{u(\lambda)u'(\xi)\} = Q\delta(\lambda-\xi) \text{ and } Q = \text{const.} \tag{2.4-6}$$

$$R_y(t,\tau) = \int_0^t \int_0^\tau H(t,\lambda)Q\delta(\lambda-\xi)H'(\tau,\xi)d\lambda d\xi \tag{2.4-7}$$

From a relation,

$$\int_a^b f(\xi)\delta(\lambda-\xi)d\xi = f(\lambda) \tag{2.4-8}$$

$$R_y(t,\tau) = \int_0^t H(t,\lambda)QH'(\tau,\lambda)d\lambda \tag{2.4-9}$$

For the process in time-invariant,

$$H(t,\tau) = H(t-\tau) \text{ for all } t,\ \tau \tag{2.4-10}$$

$$R_y(t,\tau) = \int_0^t H(t-\lambda)QH'(\tau-\lambda)d\lambda \tag{2.4-11}$$

Replacing τ by $t+\tau$,

$$R_y(t,t+\tau) = \int_0^t H(t-\lambda)QH'(t-\lambda+\tau)d\lambda = \int_0^t H(\xi)QH'(\xi+\tau)d\xi \tag{2.4-12}$$

Define $\xi = t - \lambda$,

$$\lim_{t\to\infty} R_y(t, t+\tau) = \overline{R}_y(\tau) = \int_0^\infty H(\xi) Q H'(\xi+\tau) d\xi \qquad (2.4\text{-}13)$$

- The output correlation in (2.4-13) is valid only if the dynamic system has a steady state response. If the system is not asymptotically stable (2.4-13) is not meaningful.

The expected value (mean) of the output

$$E\{y(t)\} = E\left\{\int_0^t H(t,\lambda) u(\lambda) d\lambda\right\} = \int_0^t H(t,\lambda) E\{u(\lambda)\} d\lambda \qquad (2.4\text{-}14)$$

For white noise input, $u(t)$,

$$E\{u(t)\} = 0 \qquad (2.4\text{-}15)$$

$$E\{y(t)\} = 0 \qquad (2.4\text{-}16)$$

- The response of a linear system to white noise in the steady state has a zero mean and has a correlation function given by $\int_0^\infty H(\xi) Q H'(\xi+\tau) d\xi$.

The power spectrum of the output y

$$S(\omega) = \int_{-\infty}^{\infty} \left[\int_0^\infty H(\xi) Q H'(\xi+\tau) d\xi \right] e^{-j\omega\tau} d\tau \qquad (2.4\text{-}17)$$

$$S(\omega) = \int_0^\infty H(\xi) Q \left[\int_{-\infty}^\infty H'(\xi + \tau) e^{-j\omega\tau} d\tau \right] d\xi \qquad (2.4\text{-}18)$$

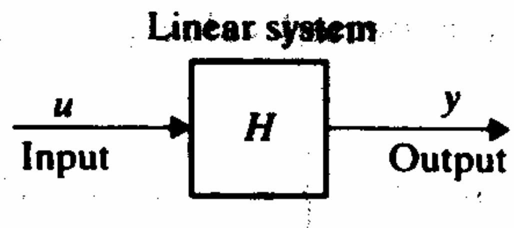
$$\int_{-\infty}^\infty H'(\xi + \tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^\infty H'(\lambda) e^{-j\omega(\lambda - \xi)} d\lambda = e^{j\omega\xi} H'(j\omega) \qquad (2.4\text{-}19)$$

$H(j\omega) = \int_{-\infty}^\infty H(t) e^{-j\omega t} dt$: the transfer function of the linear system

$$S(\omega) = \int_0^\infty H(\xi) e^{j\omega\xi} d\xi \cdot QH'(j\omega) \qquad (2.4\text{-}20)$$

$$S_y(\omega) = H(-j\omega) QH'(j\omega) \qquad (2.4\text{-}21)$$

- The spectrum of the output y of a linear system excited by white noise is the product of the transfer function matrix at negative frequency with the spectral density matrix of the white noise, with the transfer function matrix transposed.



$H(t)$ = impulse response

$H(s)$ = transfer function = $\mathscr{L}[H(t)]$

Domain	Deterministic inputs	White noise inputs
Time domain	$y(t) = \int_0^t H(t-\lambda)u(\lambda)\,d\lambda$	$R_y(\tau) = \int_0^\infty H(\xi)QH'(\xi+\tau)\,d\tau$
Frequency domain	$y(s) = H(s)u(s)$	$S_y(\omega) = H(-j\omega)QH'(j\omega)$

Figure 2.4-1 Input-Output Relation for Linear System Excited by White Noise

The most common random process after white noise is the output of a first-order low pass filter having the transfer function

$$H(s) = \frac{1}{s + \omega_0} \tag{1}$$

The impulse response corresponding to $H(s)$ is

$$h(t) = \begin{cases} e^{-\omega_0 t} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{2}$$

The correlation function of this process, often known as a first-order Markov process, is

$$r(\tau) = Q\int_0^\infty e^{-\omega_0 \xi} e^{-\omega_0(\xi+\tau)} d\xi = Qe^{-\omega_0 \tau}\int_0^\infty e^{-2\omega_0 \xi} d\xi = \frac{Q}{2\omega_0} e^{-\omega_0 \tau} \text{ for } \tau > 0 \tag{3}$$

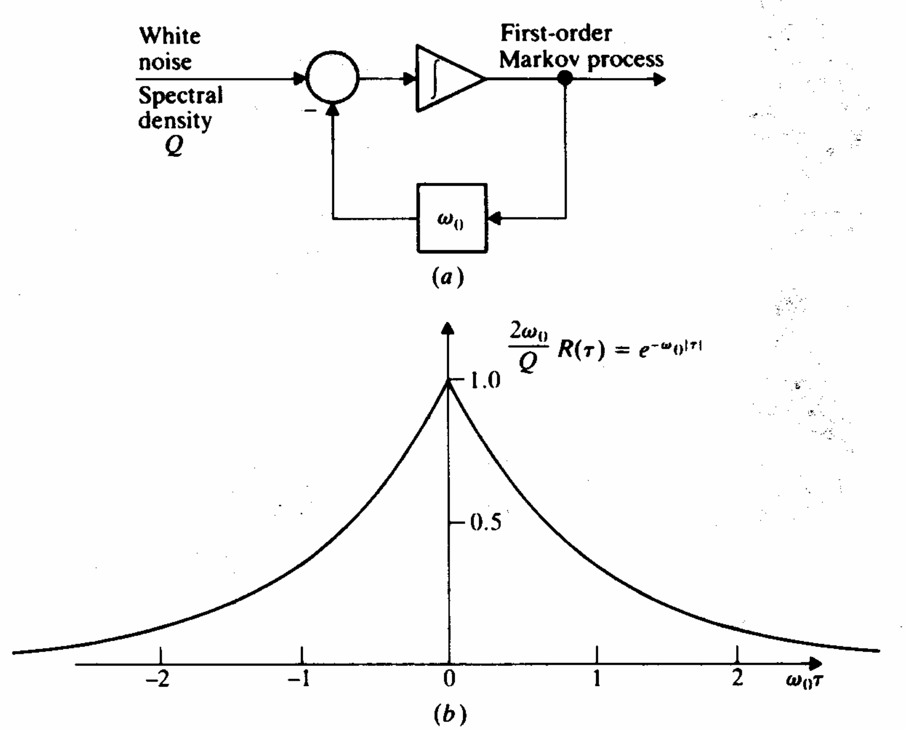
Since $h(\tau)$ is zero for $\tau < 0$, this expression is not valid for $\tau < 0$. To obtain $r(\tau)$ for negative τ we use the general relation $R(t, \tau) = R'(\tau, t)$, which in this case is

$$r(\tau) = r(-\tau) \tag{4}$$

by which (3) becomes

$$r(\tau) = \frac{Q}{2\omega_0} e^{-\omega_0 |\tau|} \tag{5}$$

as shown in the figure below.



The power spectrum is obtained either as the Fourier transform of (5) or using (2.4-21). The latter is easier.

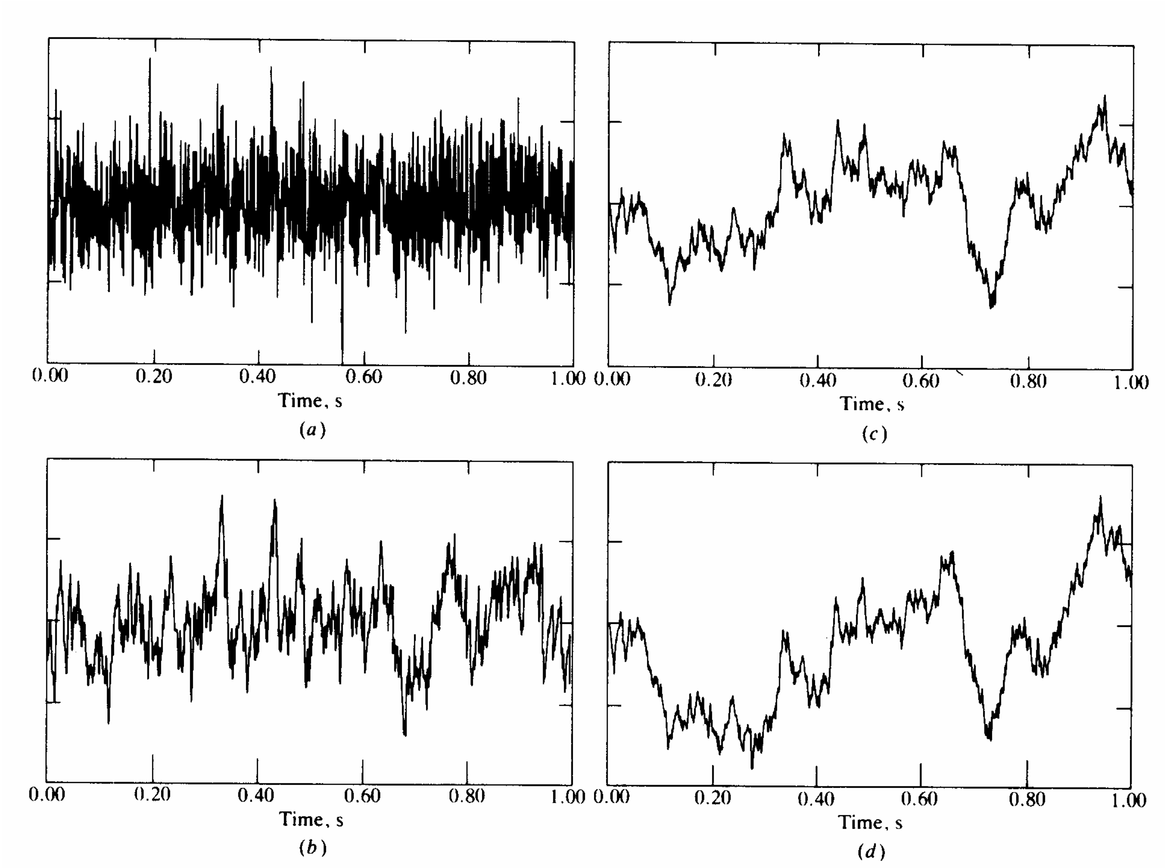
$$S(\omega) = \frac{1}{-j\omega + \omega_0} \cdot Q \cdot \frac{1}{j\omega + \omega_0} = \frac{Q}{\omega^2 + \omega_0^2} \tag{6}$$

The mean square value of the output is given by

$$r(0) = \frac{Q}{2\omega_0} \tag{7}$$

Thus the spectral density of the white noise is Q = $2\omega 0\times$(mean square value of signal). The units of the white noise spectral density Q are thus (units of the signal)2 × sec.

Figures below show (a) white noise; (b) white noise through filter with τ = 0.1 s; (c) white noise through filter with τ = 1 s; (d) white noise through filter with τ = 10 s.



2.5 Systems with State-Space Representation

The general linear system

$$\dot{x} = Ax + Bu + Fv \tag{2.5-1}$$

$$y = Cx + w \tag{2.5-2}$$

v and *w*: white-noise processes.

If the observation noise *w* and the control *u* are ignored.

$$\dot{x} = Ax + Fv \tag{2.5-3}$$

$$y = Cx \tag{2.5-4}$$

$$x(t) = \phi(t, t_0)x(t_0) + \int_{t_0}^{t} \phi(t, \lambda)F(\lambda)v(\lambda)d\lambda \tag{2.5-5}$$

t_0 : a fixed starting time

$$x(t)x'(\tau) = \phi(t, t_0)x(t_0)x'(t_0)\phi'(\tau, t_0) + x(t_0)\left[\int_{t_0}^{t} \phi(t, \lambda)F(\lambda)v(\lambda)d\lambda\right]'$$

$$+ \int_{t_0}^{t} \phi(t, \lambda)F(\lambda)v(\lambda)d\lambda \cdot x'(t_0) + \int_{t_0}^{t}\int_{t_0}^{\tau} \phi(t, \lambda)F(\lambda)v(\lambda)v'(\xi)F'(\xi)\phi'(\tau, \xi)d\xi d\lambda \tag{2.5-6}$$

Because white noise has zero mean,

$$E\left\{\int_{t_0}^{t}\phi(t,\lambda)F(\lambda)v(\lambda)d\lambda\right\}=\int_{t_0}^{t}\phi(t,\lambda)F(\lambda)E\{v(\lambda)\}d\lambda=0 \quad (2.5\text{-}7)$$

$$R_x(t,\tau)=\phi(t,t_0)E\{x(t_0)x'(t_0)\}\phi'(\tau,t_0)+\int_{t_0}^{t}\int_{t_0}^{\tau}\phi(t,\lambda)F(\lambda)E\{v(\lambda)v'(\xi)\}F'(\xi)\phi'(\tau,\xi)d\xi d\lambda \quad (2.5\text{-}8)$$

$$E\{x(t_0)x'(t_0)\}=P(t_0) \quad (2.5\text{-}9)$$

$P(t_0)$: the covariance matrix of $x(t_0)$

If v is white noise,

$$E\{v(\lambda)v'(\xi)\}=Q_v(\lambda)\delta(\lambda-\xi) \quad (2.5\text{-}10)$$

$$\int_{t_0}^{t}\phi(t,\lambda)F(\lambda)Q_v(\lambda)\left\{\int_{t_0}^{\tau}\delta(\lambda-\xi)F'(\xi)\phi'(t,\xi)d\xi\right\}d\lambda \quad (2.5\text{-}11)$$

$$\int_{t_0}^{\tau}\delta(\lambda-\xi)F'(\xi)\phi'(t,\xi)d\xi=\begin{cases}F'(\lambda)\phi'(t,\lambda) & t_0<\lambda<t\\ 0 & \text{otherwise}\end{cases} \quad (2.5\text{-}12)$$

$$R_x(t,\tau)=\phi(t,t_0)P(t_0)\phi'(\tau,t_0)+\int_{t_0}^{\bar{t}}\phi(t,\lambda)F(\lambda)Q_v(\lambda)F'(\lambda)\phi'(\tau,\lambda)d\lambda \quad (2.5\text{-}13)$$

where $\bar{t}=\min(t,\tau)$.

$$R_x(t,\tau) = P(t)\phi'(\tau,t) \text{ for } \tau \geq t \tag{2.5-14}$$

$$P(t) = R_x(t,t) = \phi(t,t_0)P(t_0)\phi'(t,t_0) + \int_{t_0}^{t} \phi(t,\lambda)F(\lambda)Q_v(\lambda)F'(\lambda)\phi'(t,\lambda)d\lambda \tag{2.5-15}$$

$P(t)$: the covariance matrix of the state $x(t)$ at time t.

- The correlation matrix $R_x(t, \tau)$, for τ not necessarily equal to t, is simply the product of covariance matrix, $P(t)$, with the transpose of the transition matrix from τ to t, when $P(t) = R_x(t, t)$.
- The matrix $P(t)$ is the solution of the matrix Riccati equation.

$$\dot{P} = AP + PA' + FQ_vF' \tag{2.5-16}$$

$$P(t)\big|_{t=t_0} = P(t_0) \tag{2.5-17}$$

- $\dot{P} = AP + PA' + FQ_vF'$, the **variance equation**, is very useful in the analysis of random processes excited by white noise, since it permits one to determine how the covariance propagates with the elapse of time, without the necessity of having to find the state-transition matrix.
- If A is a constant matrix corresponding to a stable dynamic system, and F and Q_v are constant

$$P(t) \to \bar{P} = \text{const} \tag{2.5-18}$$

$\bar{P}$: the steady state covariance matrix

$$0 = A\bar{P} + \bar{P}A' + FQ_vF' \tag{2.5-19}$$

$$y(t) = C(t)x(t) \tag{2.5-20}$$

$$y(t)y'(\tau) = C(t)x(t)x'(\tau)C'(\tau) \tag{2.5-21}$$

$$R_y(t,\tau) = E\{y(t)y'(\tau)\} = C(t)E\{x(t)x'(\tau)\}C'(\tau) = C(t)R_x(t,\tau)C'(\tau) \tag{2.5-22}$$

The covariance matrix of the output

$$P_y(t) = R_y(t,t) = C(t)P(t)C'(t) \tag{2.5-23}$$

The differential equations for the wind-turbulence process having a Dryden spectrum are

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = -\frac{1}{T^2}x_1 - \frac{2}{T}x_2 + v \tag{2}$$

and the output is given by

$$y = \frac{1}{T^2}x_1 + \frac{\sqrt{3}}{T}x_2 \tag{3}$$

The matrices representing the process are thus

$$A = \begin{bmatrix} 0 & 1 \\ -1/T^2 & -2/T \end{bmatrix}, F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1/T^2 & \sqrt{3}/T \end{bmatrix} \tag{4}$$

Let the steady state covariance matrix be

$$\overline{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \tag{5}$$

Then by $0 = A\overline{P} + \overline{P}A' + FQ_vF'$, the elements of $\overline{P}$ are given by the solutions of

$$0 = 2p_2 \tag{6}$$

$$0 = -\frac{p_1}{T^2} - \frac{2p_2}{T} + p_3 \tag{7}$$

$$0 = 2\left(-\frac{p_2}{T^2} - \frac{2p_3}{T}\right) + \sigma_z^2 T \tag{8}$$

By solving (6)-(8), thus

$$\bar{P} = \frac{\sigma_z^2 T^2}{4}\begin{bmatrix} T^2 & 0 \\ 0 & 1 \end{bmatrix} \tag{9}$$

The steady state covariance $\bar{P}_y$ of the output is obtained using $P_y(t) = C(t)P(t)C'(t)$:

$$\bar{P}_y = C\bar{P}C' = \begin{bmatrix} 1/T^2 & \sqrt{3}/T \end{bmatrix}\frac{\sigma_z^2 T^2}{4}\begin{bmatrix} T^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/T^2 \\ \sqrt{3}/T \end{bmatrix} = \sigma_z^2 \tag{10}$$

To obtain the correlation of the output we need the state-transition matrix $e^{A\tau}$ for A given in (4)

$$e^{A\tau} = \begin{bmatrix} 1+\tau/T & \tau \\ -\tau/T^2 & 1-\tau/T \end{bmatrix} \tag{11}$$

Thus, by the steady state form of $R_x(t,\tau) = P(t)\phi'(\tau,t)$,

$$R_x(\tau) = \bar{P}(e^{A\tau})' = \frac{\sigma_z^2 T^2}{4}\begin{bmatrix} T^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1+\tau/T & -\tau/T^2 \\ \tau & 1-\tau/T \end{bmatrix} = \frac{\sigma_z^2 T^2}{4}\begin{bmatrix} T^2(1+\tau/T) & -\tau \\ \tau & 1-\tau/T \end{bmatrix},\ \tau > 0 \tag{12}$$

Finally, the output correlation function is obtained by use of $R_y(t,\tau) = C(t)R_x(t,\tau)C'(\tau)$:

$$R_y(\tau) = CR_x(\tau)C' = \begin{bmatrix} 1/T^2 & \sqrt{3}/T \end{bmatrix}\frac{\sigma_z^2 T^2}{4}\begin{bmatrix} T^2(1+\tau/T) & -\tau \\ \tau & 1-\tau/T \end{bmatrix}\begin{bmatrix} 1/T^2 \\ \sqrt{3}/T \end{bmatrix} = \sigma_z^2\left(1-\frac{\tau}{2T}\right),\ \tau > 0 \tag{13}$$

And by symmetry

$$R_y(\tau) = \sigma_z^2\left(1-\frac{|\tau|}{2T}\right) \tag{14}$$

3 Kalman Filters: Optimal Observers

3.1 Kalman Filter Problem

A dynamic process

$$\dot{x} = Ax + Bu + Fv \quad (3.1\text{-}1)$$

$$y = Cx + w \quad (3.1\text{-}2)$$

v and w : white noise processes, The optimal state estimator

$$\dot{\hat{x}} = A\hat{x} + Bu + \hat{K}(y - C\hat{x}) \quad (3.1\text{-}3)$$

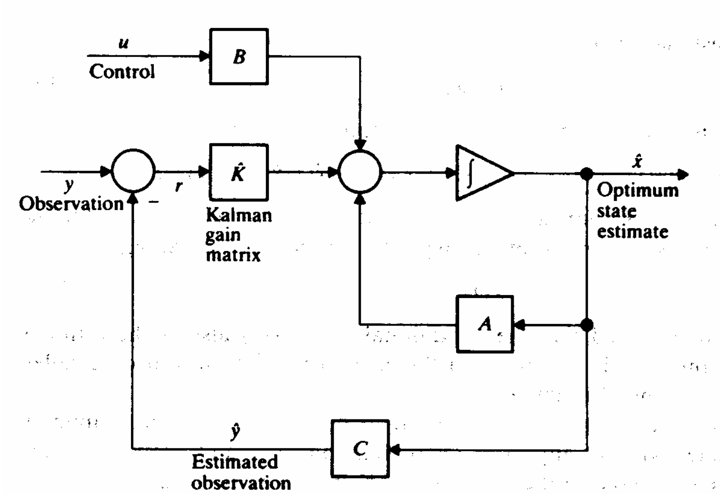


Figure 3.1-1 Kalman Filter is an Optimum Observer

- The random processes of disturbance input and measurement noise in Kalman filter are assumed white and gaussian.
- The gaussian requirement is a condition on the first-order probability density functions of w and v.

$$pdf(w) = \frac{1}{(2\pi)^{n/2}|w|^{1/2}} \exp\left\{-\frac{1}{2} w' W^{-1} w\right\} \tag{3.1-4}$$

- If v and w are white and Gaussian random processes, then as proved by Kalman and Bucy, the Kalman filter is the best of all possible filters. There is no other filter, linear or nonlinear, better than the linear Kalman filter.
- Kalman defined the state estimate $\hat{x}(t)$ as the conditional mean of $x(t)$, given the observation data $y(\tau)$ for $\tau \le t$.

$$\hat{x}(t) = E\{x(t) \mid y(\tau), \tau \le t\} \tag{3.1-5}$$

$$e(t) = x(t) - \bar{x}(t) \tag{3.1-6}$$

$\bar{x}(t)$: any estimate of $x(t)$

$$e(t)e'(t) = [x(t) - \bar{x}(t)][x'(t) - \bar{x}'(t)] = x(t)x'(t) - \bar{x}(t)x'(t) - x(t)\bar{x}'(t) + \bar{x}(t)\bar{x}'(t) \tag{3.1-7}$$

$$E\{e(t)e'(t) \mid y(\tau), \tau \le t\} = E\{x(t)x'(t)\} - \bar{x}(t)\hat{x}'(t) - \hat{x}(t)\bar{x}'(t) + \bar{x}(t)\bar{x}'(t) \tag{3.1-8}$$

$$\bar{x}(t) = \hat{x}(t) + \zeta(t) \tag{3.1-9}$$

$$E\{e(t)e'(t) \mid y(\tau), \tau \le t\} = E\{x(t)x'(t)\} - \hat{x}(t)\hat{x}'(t) + \zeta(t)\zeta'(t) \tag{3.1-10}$$

- Since $\zeta(t)\zeta'(t)$ is a nonnegative quantity, the conditional covariance matrix is minimized by setting $\zeta(t) = 0$.

$$\bar{x}(t) = \hat{x}(t) \tag{3.1-11}$$

- The conditional mean, $\hat{x}(t)$, is the estimate that minimizes the covariance matrix of the error.

3.2 Kalman Filter Gain and Variance Equations

The error

$$e = x - \hat{x} \tag{3.2-1}$$

The differential equation for the error

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Fv - A\hat{x} - K(Cx + w - C\hat{x}) = (A - KC)e + Fv - Kw \tag{3.2-2}$$

v and *w*: white noise processes

$$\xi = Fv - Kw \tag{3.2-3}$$

- $\xi = Fv - Kw$ is also white noise, with a covariance matrix Q_ξ.

$$E\{\xi(t)\xi'(\tau)\} = F(t)E\{v(t)v'(\tau)\}F'(\tau) - K(t)E\{w(t)v'(\tau)\}F'(\tau) - F(t)E\{v(t)w'(\tau)\}K'(\tau) + K(t)E\{w(t)w'(\tau)\}K'(\tau) \tag{3.2-4}$$

$$E\{v(t)v'(\tau)\} = V(t)\delta(t - \tau) \tag{3.2-5}$$

$$E\{v(t)w'(\tau)\} = X(t)\delta(t - \tau) \tag{3.2-6}$$

$$E\{w(t)w'(\tau)\} = W(t)\delta(t - \tau) \tag{3.2-7}$$

$$E\{\xi(t)\xi'(\tau)\} = Q_\xi(t)\delta(t - \tau) \tag{3.2-8}$$

$$Q_\xi(t) = F(t)V(t)F'(t) - K(t)X'(t)F'(t) - F(t)X(t)K'(t) + K(t)W(t)K'(t) \tag{3.2-9}$$

The differential equation of a linear system excited by white noise ξ

$$\dot{e} = (A - KC)e + \xi \tag{3.2-10}$$

The variance equation

$$\dot{P} = (A - KC)P + P(A' - C'K') + Q_{\xi} = (A - KC)P + P(A' - C'K') + FVF' - KX'F' - FXK' + KWK' \qquad (3.2\text{-}11)$$

P: the covariance matrix of the error

- If the cross-covariance X between the excitation noise v and the observation noise w were absent ($X = 0$), (3.2-11) would have the same form as the optimal control equation and we would be able to write the solution for the optimum gain matrix.

$$\hat{K} = \hat{P}C'W^{-1} \qquad (3.2\text{-}12)$$

- The optimizing covariance matrix is given by the matrix Riccati equation.

$$\dot{\hat{P}} = A\hat{P} + \hat{P}A' - \hat{P}C'W^{-1}C\hat{P} + FVF' \qquad (3.2\text{-}13)$$

For non-zero cross-correlation matrix ($X \neq 0$),

$$P = \hat{P} + U \qquad (3.2\text{-}14)$$

$$K = \hat{K} + \Gamma \qquad (3.2\text{-}15)$$

$\hat{P}$ and $\hat{K}$: the optimum covariance matrix and observer gain matrix

$$\dot{\hat{P}} + \dot{U} = (A - \hat{K}C - \Gamma C)(\hat{P} + U) + (\hat{P} + U)(A' - C'K' - C'\Gamma') + FVF' - (\hat{K} + \Gamma)X'F' - FX(\hat{K}' + \Gamma') + (\hat{K} + \Gamma)W(\hat{K}' + \Gamma') \qquad (3.2\text{-}16)$$

$$\dot{\hat{P}} = (A - \hat{K}C)\hat{P} + \hat{P}(A' - C'K') + FVF' - \hat{K}X'F' - FX\hat{K}' + \hat{K}W\hat{K}' \qquad (3.2\text{-}17)$$

$$\dot{U} = (A - \hat{K}C - \Gamma C)U + U(A' - C'K' - C'\Gamma') + +\Gamma W\Gamma' + \Gamma(W\hat{K}' - C\hat{P} - X'F') + (\hat{K}W - \hat{P}C' - FX)\Gamma' \qquad (3.2\text{-}18)$$

- If $\hat{P}$ is the minimum covariance matrix P must be greater than $\hat{P}$ for any choice of Γ.
- U must be positive semidefinite.
- U can be made negative definite by suitable choice of Γ unless the coefficient of Γ vanishes entirely, in which case U will be positive semidefinite.

$$\hat{K}W = \hat{P}C' + FX \tag{3.2-19}$$

For nonsingular observation noise spectral density matrix, W,

$$\hat{K} = (\hat{P}C' + FX)W^{-1} \tag{3.2-20}$$

The matrix Riccati equation

$$\dot{\hat{P}} = \tilde{A}\hat{P} + \hat{P}\tilde{A}' - \hat{P}C'W^{-1}C\hat{P} + F\tilde{V}F' \tag{3.2-21}$$

$$\tilde{A} = A - FXW^{-1}C \text{ and } \tilde{V} = V - XW^{-1}X' \tag{3.2-22}$$

3.3 Steady-State Kalman Filter

- The matrix Riccati equation is valid for any finite time interval.
- If time is infinite, the solutions may tend to infinity or they may remain finite.
- If all the matrices on the right-hand side of the matrix Riccati equation are constant, then a constant, steady state solution may exist, the solution of the matrix quadratic equation, the algebraic Riccati equation (ARE).

$$0 = \tilde{A}\hat{P} + \hat{P}\tilde{A}' - \hat{P}C'W^{-1}C\hat{P} + F\tilde{V}F' \qquad (3.3\text{-}1)$$

The ARE has a unique positive definite solution if either

(a) The system is asymptotically stable, or

(b) The system defined by the pair $[A, C]$ is observable and the system defined by the pair $[A, FV^{1/2}]$ is controllable.

If it were possible to balance an inverted pendulum, it would not remain balanced without control owing to the inevitable presence of various types of disturbances, such as random air currents. Thus, if the accelerations due to the disturbances are represented by v, the differential equations for the pendulum are

$$\dot{\theta} = \omega \tag{1}$$

$$\dot{\omega} = \Omega^2 \theta + u + v \tag{2}$$

where u is the control acceleration and v is the disturbance acceleration. Assume the DC motor's back emf is small and negligible. The matrices corresponding to (1) and (2) are

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ F = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3}$$

If the quantity observed is the position $x_1 = \theta$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + w \tag{4}$$

Hence

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{5}$$

Let the optimum covariance matrix be

$$\hat{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \tag{6}$$

Then, by $0 = A\hat{P} + \hat{P}A' - \hat{P}C'W^{-1}C\hat{P} + FVF'$, the elements of $\hat{P}$ satisfy

$$0 = 2p_2 - \frac{p_1^2}{W} \tag{7}$$

$$0 = p_3 + \Omega^2 p_1 - \frac{p_1 p_2}{W} \tag{8}$$

$$0 = 2\Omega^2 p_2 - \frac{p_2^2}{W} + V \tag{9}$$

where V and W are the spectral density (1×1) matrices of the excitation noise and observation noise, respectively. The solutions are

$$p_1 = \Omega W \sqrt{2\gamma} \tag{10}$$

$$p_2 = \Omega^2 W \gamma \tag{11}$$

$$p_3 = \Omega^3 W \sqrt{2\gamma}(\gamma - 1) \tag{12}$$

where

$$\gamma = 1 + \sqrt{1 + \frac{V}{\Omega^4 W}} \tag{13}$$

The Kalman filter gain is determined.

$$\hat{K} = \hat{P}C'W^{-1} = \begin{bmatrix} \Omega\sqrt{2\gamma} \\ \Omega^2\gamma \end{bmatrix} \tag{14}$$

The closed-loop filter poles and transfer functions from the measured angle y to the estimated state $\hat{x}_1 = \hat{\theta}$ and $\hat{x}_2 = \hat{\omega}$ are of interest. Assuming that the input u is zero we have

$$s\hat{x}(s) = A\hat{x}(s) + \hat{K}[y(s) - C\hat{x}(s)] = (A - \hat{K}C)\hat{x}(s) + \hat{K}y(s) \tag{15}$$

$$\hat{x}(s) = (sI - A + \hat{K}C)^{-1}\hat{K}y(s) = (sI - A_0)^{-1}\hat{K}y(s) \tag{16}$$

In this example,

$$A_0 = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} - \begin{bmatrix} \Omega\sqrt{2\gamma} \\ \Omega^2\gamma \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -\Omega\sqrt{2\gamma} & 1 \\ \Omega^2(1-\gamma) & 0 \end{bmatrix} \tag{17}$$

Thus

$$(sI - A_0)^{-1} = \begin{bmatrix} s + \Omega\sqrt{2\gamma} & -1 \\ \Omega^2(\gamma - 1) & s \end{bmatrix}^{-1} = \frac{1}{\Delta(s)}\begin{bmatrix} s & 1 \\ \Omega^2(1-\gamma) & s + \Omega\sqrt{2\gamma} \end{bmatrix} \tag{18}$$

where $\Delta(s)$ is the closed-loop characteristic polynomial, given by

$$\Delta(s) = s^2 + \Omega\sqrt{2\gamma}s + \Omega^2(\gamma - 1) \tag{19}$$

Thus,

$$\hat{x}(s) = \begin{bmatrix} \hat{\theta}(s) \\ \hat{\omega}(s) \end{bmatrix} = \frac{1}{\Delta(s)}\begin{bmatrix} \Omega(s\sqrt{2\gamma} + \Omega\gamma) \\ \Omega^2(\gamma s + \Omega\sqrt{2\gamma}) \end{bmatrix} y(s) \tag{20}$$

In particular

$$H_1(s) = \frac{\hat{\theta}(s)}{y(s)} = \frac{\Omega(s\sqrt{2\gamma} + \Omega\gamma)}{s^2 + \Omega\sqrt{2\gamma}s + \Omega^2(\gamma - 1)} \tag{21}$$

$$H_2(s) = \frac{\hat{\omega}(s)}{y(s)} = \frac{\Omega^2(\gamma s + \Omega\sqrt{2\gamma})}{s^2 + \Omega\sqrt{2\gamma}s + \Omega^2(\gamma - 1)} \tag{22}$$

The closed-loop poles of the filter are given by

$$s = -\Omega\left(\sqrt{\frac{\gamma}{2}} \pm j\sqrt{\frac{\gamma}{2} - 1}\right) \tag{23}$$

The zeros of the filters $H_1(s)$ and $H_2(s)$, respectively, lie at

$$s = -\Omega\sqrt{\frac{\gamma}{2}} \text{ for angular position} \tag{24}$$

$$s = -\Omega\sqrt{\frac{2}{\gamma}} \text{ for angular velocity} \tag{25}$$

As the excitation noise covariance matrix V tends to zero, γ, as given by (13), approaches 2

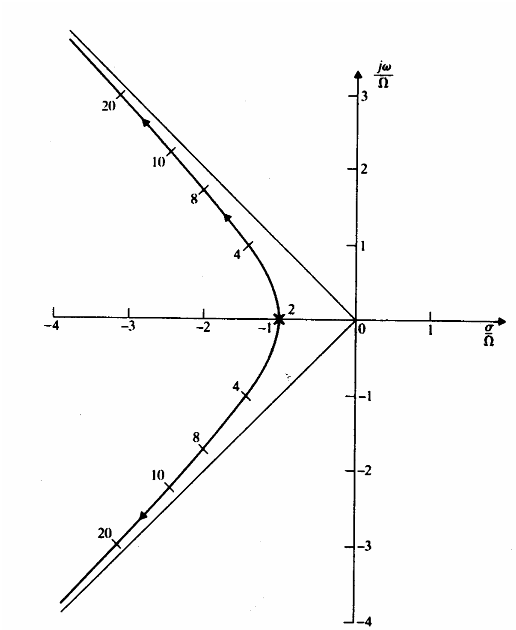
$$H_1(s) \to \frac{2\Omega}{s + \Omega} \tag{26}$$

$$H_2(s) \to \frac{2\Omega^2}{s + \Omega} \tag{27}$$

In other words the zero of the numerator tends to one pole of the denominator, and both filters become first-order. Note also that the optimum estimate of the angular velocity is simply the natural frequency Ω times the angular position. As the excitation noise covariance matrix tends to infinity (or as the observation noise covariance matrix tends to zero) γ tends to infinity and the closed-loop poles of the observer

$$s = -\Omega\sqrt{\frac{\gamma}{2}}(1 \pm j) \tag{28}$$

which are lines at 45° angles from the real axis. The next figure shows poles of Kalman filter for inverted pendulum



The differential equations for the proof mass, with the acceleration due to the capture force are

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = u + a \tag{2}$$

The position of the proof mass is determined by some sort of "pick-off"; e.g., magnetic or optical. The output of the pick-off is

$$y = x_1 + w \tag{3}$$

where w is the pick-off noise which we assume to be white.

We assume that the acceleration a is a Wiener process

$$\dot{a} = v \tag{4}$$

where v is white noise with spectral density matrix V. If V were zero then (4) would become $\dot{a} = 0$, that is, a would be an unknown constant. But as we will soon see, it is necessary to assume $V \neq 0$ in order to get a meaningful filter design.

Represent a by another state variable x_3 and adjoin (4) to (1), (2), and (3):

$$\dot{x}_1 = x_2 \tag{5}$$

$$\dot{x}_2 = u + x_3 \tag{6}$$

$$\dot{x}_3 = v \tag{7}$$

For this system the defining matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ F = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad (8)$$

Let the optimum covariance matrix be

$$\hat{P} = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix} \quad (9)$$

Then the components of $\hat{P}$ satisfy $\dot{\hat{P}} = 0 = A\hat{P} + \hat{P}A' - \hat{P}C'W^{-1}C\hat{P} + FVF'$.

$$\dot{p}_1 = 0 = 2p_2 - \frac{p_1^2}{W} \quad (10)$$

$$\dot{p}_2 = 0 = p_3 + p_4 - \frac{p_1 p_2}{W} \quad (11)$$

$$\dot{p}_3 = 0 = p_5 - \frac{p_1 p_3}{W} \quad (12)$$

$$\dot{p}_4 = 0 = 2p_5 - \frac{p_2^2}{W} \quad (13)$$

$$\dot{p}_5 = 0 = p_6 - \frac{p_2 p_3}{W} \quad (14)$$

$$\dot{p}_6 = 0 = -\frac{p_3^2}{W} + V \quad (15)$$

These can be solved readily. The resulting solutions are

$$p_1 = 2V^{1/6}W^{5/6},\ p_2 = 2V^{1/3}W^{1/3},\ p_3 = V^{1/2}W^{1/2},\ p_4 = 3V^{1/2}W^{1/2},\ p_5 = 2V^{2/3}W^{1/3},\ p_6 = 2V^{5/6}W^{1/6} \tag{16}$$

We can now compute the Kalman filter gain matrix

$$\hat{K} = \hat{P}C'W^{-1} = \begin{bmatrix} p_1/W \\ p_2/W \\ p_3/W \end{bmatrix} = \begin{bmatrix} 2(V/W)^{1/6} \\ 2(V/W)^{1/3} \\ (V/W)^{1/2} \end{bmatrix} \tag{17}$$

$$A_0 = A - KC = \begin{bmatrix} -2\overline{\Omega} & 1 & 0 \\ -2\overline{\Omega}^2 & 0 & 1 \\ -\overline{\Omega}^3 & 0 & 0 \end{bmatrix} \tag{18}$$

where

$$\overline{\Omega} = \left(\frac{V}{W}\right)^{1/6} \tag{19}$$

Thus the filter characteristic equation is

$$|sI - A_0| = \begin{vmatrix} s + 2\overline{\Omega} & -1 & 0 \\ 2\overline{\Omega}^2 & s & -1 \\ \overline{\Omega}^3 & 0 & s \end{vmatrix} = s^3 + 2\overline{\Omega}s^2 + 2\overline{\Omega}^2 s + \overline{\Omega}^3 = 0 \tag{20}$$

The characteristic roots are

$$s_1 = -\overline{\Omega}, \quad s_2, s_3 = -\overline{\Omega}\left(\frac{1}{2} \pm j\sqrt{\frac{3}{2}}\right) \tag{21}$$

4 Linear Quadratic Gaussian Control: The Separation Theorem

4.1 The Separation Theorem

To minimize the expected error in controlling a linear system,

$$\dot{x} = Ax + Bu + Fv \quad (4.1\text{-}1)$$

with observations

$$y = Cx + w \quad (4.1\text{-}2)$$

(a) Use the control law

$$u = -\hat{G}\hat{x} \quad (4.1\text{-}3)$$

where $\hat{x}$ is the output of a linear observer

$$\dot{\hat{x}} = A\hat{x} + Bu + \hat{K}(y - C\hat{x}) \quad (4.1\text{-}4)$$

(b) Find the control gain matrix $\hat{G}$ as the solution of the corresponding deterministic optimal control problem.

(c) Find the observer gain matrix $\hat{K}$ as the optimum gain for the corresponding Kalman filter.

The differential equations for the proof mass, with the acceleration due to the capture force are

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = u + a \tag{2}$$

where x_1 and x_2 are the position and velocity of the proof mass, a is the external acceleration and u is the control input. By LQR, the control input is determined to minimize $V = \int_t^{\infty} (x_1^2 + \frac{u^2}{c^2}) d\tau$, and finally given by

$$u = -g_1\hat{x}_1 - g_2\hat{x}_2 - g_a\hat{a} = -c\hat{x}_1 - \sqrt{2c}\,\hat{x}_2 - \hat{a} \tag{3}$$

where c is the reciprocal of the control weighting and may be regarded as one of the design parameters.

The Kalman filter is determined and finally given by

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1(y - \hat{x}_1) = \hat{x}_2 + 2(V/W)^{1/6}(y - \hat{x}_1) \tag{4}$$

$$\dot{\hat{x}}_2 = \hat{a} + u + k_2(y - \hat{x}_1) = \hat{a} + u + 2(V/W)^{1/3}(y - \hat{x}_1) \tag{5}$$

$$\dot{\hat{a}} = k_3(y - \hat{x}_1) = (V/W)^{1/2}(y - \hat{x}_1) \tag{6}$$

where V is the spectral density of the acceleration rate to be measured, and W is the spectral density of the noise in measuring the pick-off position.

By the separation principle, the (nominal) closed-loop pole locations are the roots of the characteristic polynomial for full-state feedback and the roots of the characteristic polynomial of the Kalman filter. The former are at

$$s = -\frac{\sqrt{2c}}{2} \pm \frac{\sqrt{2c}}{2} j \tag{7}$$

and the latter are at

$$s = -\overline{\Omega} \text{ and } s = -\overline{\Omega}\left(\frac{1}{2} \pm j\sqrt{\frac{3}{2}}\right) \tag{8}$$