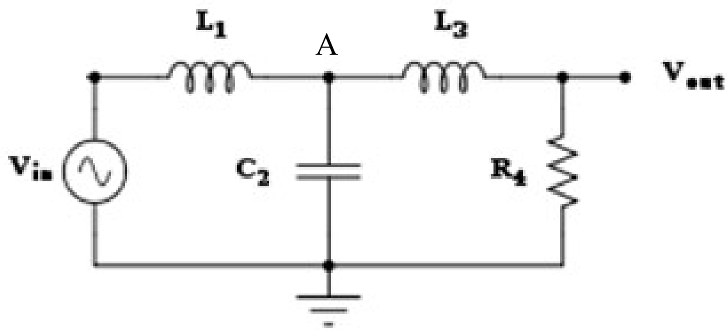
Midterm Examination Sensing and Actuation AT74.03 March 9, 2021

Time: 10:00-12:00 h. Open Book
Marks: 100

Attempt all questions.

Q.1 A third-order low-pass Butterworth passive filter is shown by the circuit below.



(a) By Thevenin's theorem, prove that the transfer function of this circuit is expressed by (10)

$$\frac{V_{out}}{V_{in}} = \frac{R_4}{s^3(L_1C_2L_3) + s^2(L_1C_2R_4) + s(L_1 + L_3) + R_4}$$

(b) Determine 3-db cut-off frequency of this low-pass filter when $L_1 = \frac{3}{2}H, C_2 = \frac{4}{3}F, L_3 = \frac{1}{2}H, R_4 = 1\Omega$. (10)

Solution

(a)

The voltage at point A is determined.

$$V_A = \frac{V_{in}}{sL_1 + \frac{1}{sC_2}} \times \frac{1}{sC_2} = \frac{V_{in}}{s^2L_1C_2 + 1}$$

The impedance at point A is determined.

$$Z_A = \frac{sL_1 \times \frac{1}{sC_2}}{sL_1 + \frac{1}{sC_2}} = \frac{sL_1}{s^2L_1C_2 + 1}$$

The voltage output is determined.

$$V_{out} = \frac{V_A}{Z_A + sL_3 + R_4} \times R_4 = \frac{V_{in}R_4}{(s^2L_1C_2 + 1)\left(\frac{sL_1}{s^2L_1C_2+1} + sL_3 + R_4\right)}$$

$$\frac{V_{out}}{V_{in}} = \frac{R_4}{s^3(L_1C_2L_3) + s^2(L_1C_2R_4) + s(L_1 + L_3) + R_4}$$

(b)

Substitute all the parameters into the transfer function.

$$\frac{V_{out}}{V_{in}} = \frac{R_4}{s^3(L_1C_2L_3) + s^2(L_1C_2R_4) + s(L_1 + L_3) + R_4} = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

Substitute $s = \omega j$.

$$\frac{V_{out}}{V_{in}} = \frac{1}{-\omega^3 j - 2\omega^2 + 2\omega j + 1} = \frac{1}{(1 - 2\omega^2) + (-\omega^3 + 2\omega)j}$$

Determine the magnitude.

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{(1 - 2\omega^2)^2 + (-\omega^3 + 2\omega)^2}}$$

Determine 3.01 db Cut-off frequency.

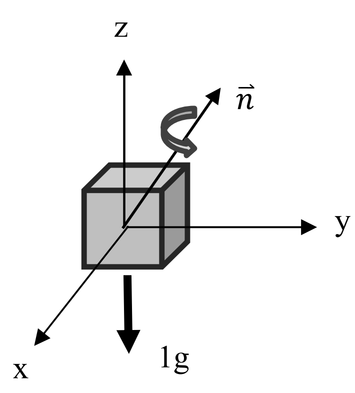
$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{(1 - 2\omega^2)^2 + (-\omega^3 + 2\omega)^2}}$$

$$2 = 1 - 4\omega^2 + 4\omega^4 + \omega^6 - 4\omega^4 + 4\omega^2 = 1 + \omega^6$$

$$1 = \omega^6$$

$$\omega = 1\, rad/s$$

Q.2 A 3-axis accelerometer is used to determine the attitude of an object rotating freely at the sea level. Assume the accelerometers are identical in all axis, the seismic mass deflects 1 mm from the acceleration of 1g $(9.8\, ms^{-2})$. Determine the deflections of seismic mass in all axes of the accelerometer when the object rotates around a normal vector $\vec{n} = \frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k$ for 30° as shown in the figure below. (20)



<u>**Solution**</u>

Determine new coordinate of $(\acute{x}, \acute{y}, \acute{z})$ from the old coordinate of $(x, y, z) = (i, j, k)$.

The new coordinate can be determined by any methods; for examples, using vector consideration, using Rodrigues formula, using Quarternion formula, etc.

By Rodrigues formula,

$$\acute{x} = \left(1 - cos(\theta)\right)(x \cdot \vec{n})\vec{n} + cos(\theta)x + sin(\theta)(\vec{n} \times x)$$

$$\acute{x} = \left(1 - cos(30°)\right)\left(i \cdot \left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right)\right)\left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right) + cos(30°)i$$

$$+ sin(30°)\left(\left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right) \times i\right)$$

$$\acute{x} = \left(\frac{2-\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{14}}\right)\left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right) + \left(\frac{\sqrt{3}}{2}\right)i + \left(\frac{1}{2}\right)\left(\frac{3}{\sqrt{14}}j - \frac{2}{\sqrt{14}}k\right)$$

$$\acute{x} = (0.8756i + 0.4200j - 0.2386k)$$

Thus, the deflection of seismic mass along x axis from gravity acceleration is 0.2386 mm.

$$\acute{y} = \left(1 - cos(\theta)\right)(y \cdot \vec{n})\vec{n} + cos(\theta)y + sin(\theta)(\vec{n} \times y)$$

$$\acute{y} = \left(1 - cos(30°)\right)\left(j \cdot \left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right)\right)\left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right) + cos(30°)j$$

$$+ sin(30°)\left(\left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right) \times j\right)$$

$$\acute{x} = \left(\frac{2-\sqrt{3}}{2}\right)\left(\frac{2}{\sqrt{14}}\right)\left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right) + \left(\frac{\sqrt{3}}{2}\right)j + \left(\frac{1}{2}\right)\left(-\frac{3}{\sqrt{14}}i + \frac{1}{\sqrt{14}}k\right)$$

$$\acute{y} = (-0.3818i + 0.9043j + 0.1910k)$$

Thus, the deflection of seismic mass along y axis from gravity acceleration is -0.1910 mm.

$$\acute{z} = \left(1 - cos(\theta)\right)(z \cdot \vec{n})\vec{n} + cos(\theta)z + sin(\theta)(\vec{n} \times z)$$

$$\acute{z} = \left(1 - cos(30°)\right)\left(k \cdot \left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right)\right)\left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right) + cos(30°)k$$

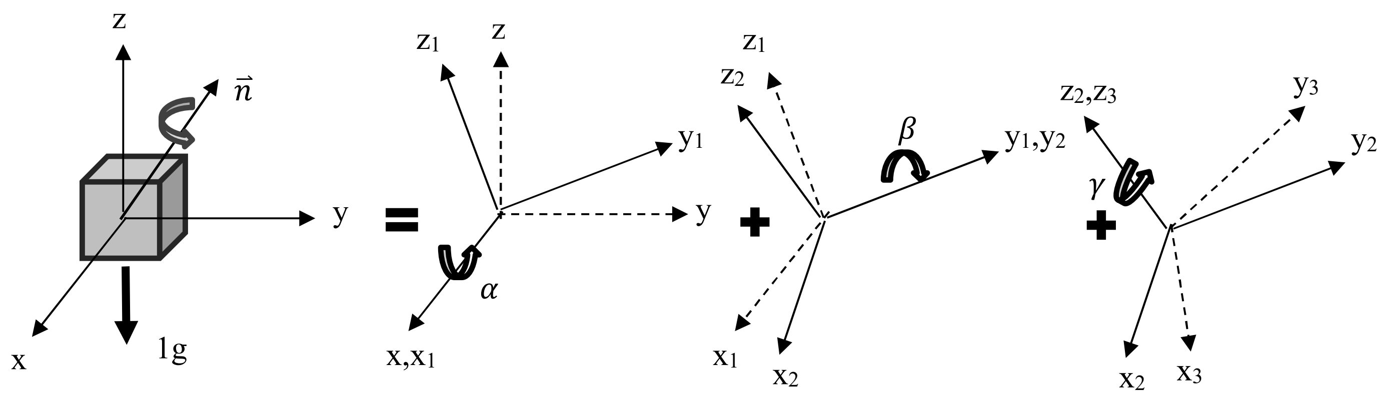
$$+ sin(30°)\left(\left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right) \times k\right)$$

$$\acute{z} = \left(\frac{2-\sqrt{3}}{2}\right)\left(\frac{3}{\sqrt{14}}\right)\left(\frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k\right) + \left(\frac{\sqrt{3}}{2}\right)k + \left(\frac{1}{2}\right)\left(\frac{2}{\sqrt{14}}i - \frac{1}{\sqrt{14}}j\right)$$

$$\acute{z} = (0.2960i - 0.0762j + 0.9522k)$$

Thus, the deflection of seismic mass along z axis from gravity acceleration is -0.9522 mm.

Q.3 If 3 single-axis rate-integrating gyros are used to measure rotations around the Euler coordinate of (x, y, z) or (roll (α), pitch (β), and yaw (γ)) of the object in **Q.2**. Assume all the gyros are identical, the gyro deflects 1° from the angular displacement of 10°. Determine the deflections of the gyros in successive rotations of roll, pitch, and yaw when the object rotates around a normal vector $\vec{n} = \frac{1}{\sqrt{14}}i + \frac{2}{\sqrt{14}}j + \frac{3}{\sqrt{14}}k$ for 30° as shown in the figure below. (20)



Solution

Rotation matrices are determined.

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\alpha) & -sin(\alpha) \\ 0 & sin(\alpha) & cos(\alpha) \end{bmatrix}$$

$$R_y = \begin{bmatrix} cos(\beta) & 0 & sin(\beta) \\ 0 & 1 & 0 \\ -sin(\beta) & 0 & cos(\beta) \end{bmatrix}$$

$$R_z = \begin{bmatrix} cos(\gamma) & -sin(\gamma) & 0 \\ sin(\gamma) & cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotations in three axis.

$$R = R_x R_y R_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\alpha) & -sin(\alpha) \\ 0 & sin(\alpha) & cos(\alpha) \end{bmatrix} \begin{bmatrix} cos(\beta) & 0 & sin(\beta) \\ 0 & 1 & 0 \\ -sin(\beta) & 0 & cos(\beta) \end{bmatrix} \begin{bmatrix} cos(\gamma) & -sin(\gamma) & 0 \\ sin(\gamma) & cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} cos(\beta)cos(\gamma) & -cos(\beta)sin(\gamma) & sin(\beta) \\ -sin(\alpha)sin(\beta)cos(\gamma) + cos(\alpha)sin(\gamma) & sin(\alpha)sin(\beta)sin(\gamma) + cos(\alpha)cos(\gamma) & -sin(\alpha)cos(\beta) \\ -cos(\alpha)sin(\beta)cos(\gamma) + sin(\alpha)sin(\gamma) & cos(\alpha)sin(\beta)sin(\gamma) + sin(\alpha)cos(\gamma) & cos(\alpha)cos(\beta) \end{bmatrix}$$

$$[\acute{x} \quad \acute{y} \quad \acute{z}] = R$$

$$\begin{bmatrix} 0.8756 & -0.3818 & 0.2960 \\ 0.4200 & 0.9043 & -0.0762 \\ -0.2386 & 0.1910 & 0.9522 \end{bmatrix}$$

$$= \begin{bmatrix} cos(\beta)cos(\gamma) & -cos(\beta)sin(\gamma) & sin(\beta) \\ -sin(\alpha)sin(\beta)cos(\gamma) + cos(\alpha)sin(\gamma) & sin(\alpha)sin(\beta)sin(\gamma) + cos(\alpha)cos(\gamma) & -sin(\alpha)cos(\beta) \\ -cos(\alpha)sin(\beta)cos(\gamma) + sin(\alpha)sin(\gamma) & cos(\alpha)sin(\beta)sin(\gamma) + sin(\alpha)cos(\gamma) & cos(\alpha)cos(\beta) \end{bmatrix}$$

Consider element (1,3),

$$0.2960 = sin(\beta)$$

$$\beta = 0.3005\,rad = 17.2175°$$

Thus the gyro used to detect pitch angle deflects 1.7218°.

Divide element (2,3) by element (3,3),

$$\frac{0.0762}{0.9522} = tan(\alpha)$$

$$\alpha = 0.0799\,rad = 4.5754°$$

Thus the gyro used to detect roll angle deflects 0.4575°.

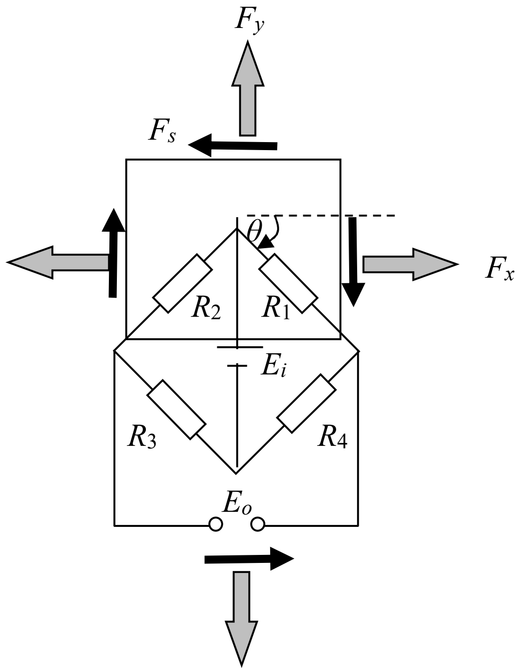
Divide element (1,2) by element (1,1),

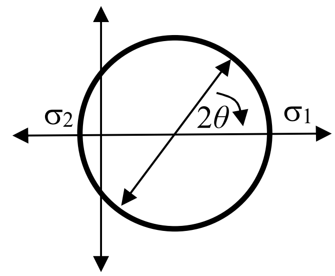
$$\frac{0.3818}{0.8756} = tan(\gamma)$$

$$\gamma = 0.4112\,rad = 23.5593°$$

Thus the gyro used to detect yaw angle deflects 2.3559°.

Q.4 4 identical strain gages; 2 gages, R_1 and R_2, attached to a cubic block of a material and 2 gages, R_3 and R_4, floated as shown below, are used to determine the principal normal stresses when a two axial forces, F_x, F_y and shearing force, F_s are applied to the block as shown in the figure below. It's Mohr's circle is also shown in the same figure. Assume each side the cubic block has length of 10 cm and made of the material with Young's modulus, E, 150 GPa and Poisson's ratio, ν, 0.25. The 4 identical strain gages, each has the nominal resistance, R, 120 Ω with gage factor, S_g, of 3. E_i is 5 V. If axial forces (F_x, F_y) and shearing force (F_s) have the magnitude of 500,000, 300,000, and 400,000 N respectively are applied to the block, determine the alignment angle, θ, of strain gage R_1 that can detect the principal normal stress, σ_1. Strain gage R_2 is always aligned 90° relative to strain gage R_1 and used to detect σ_2. What is the voltage output reading? (20)





__**Solution**__

Axial stress along x-axis,

$$\sigma_x = \frac{F_x}{A_x} = \frac{500{,}000}{0.01} = 50{,}000{,}000\ Pa = 50\ MPa$$

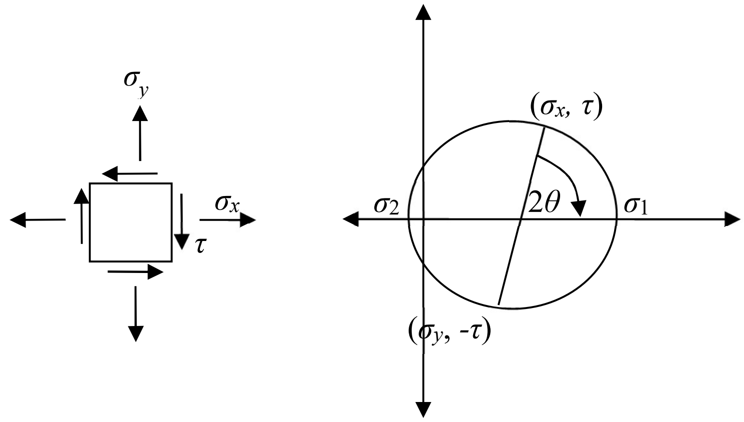
Axial stress along y-axis,

$$\sigma_y = \frac{F_y}{A_y} = \frac{300{,}000}{0.01} = 30{,}000{,}000\ Pa = 30\ MPa$$

Shearing stress,

$$\tau = \frac{F_s}{A_s} = \frac{400{,}000}{0.01} = 40{,}000{,}000\ Pa = 40\ MPa$$

Cross section and Mohr's circle,



Center of the Mohr's circle,

$$\frac{\sigma_x + \sigma_y}{2} = 40\ MPa$$

Radius of the Mohr's circle,

$$\sqrt{10^2 + 40^2} = 41.23\ MPa$$

$$\sigma_1 = 40 + 41.23 = 81.23\ MPa$$

$$\sigma_2 = 40 - 41.23 = -1.23\ MPa$$

$$2\theta = atan\left(\frac{40}{10}\right) = 75.96°$$

The alignment of strain gage R_1,

$$\theta = 37.98°$$

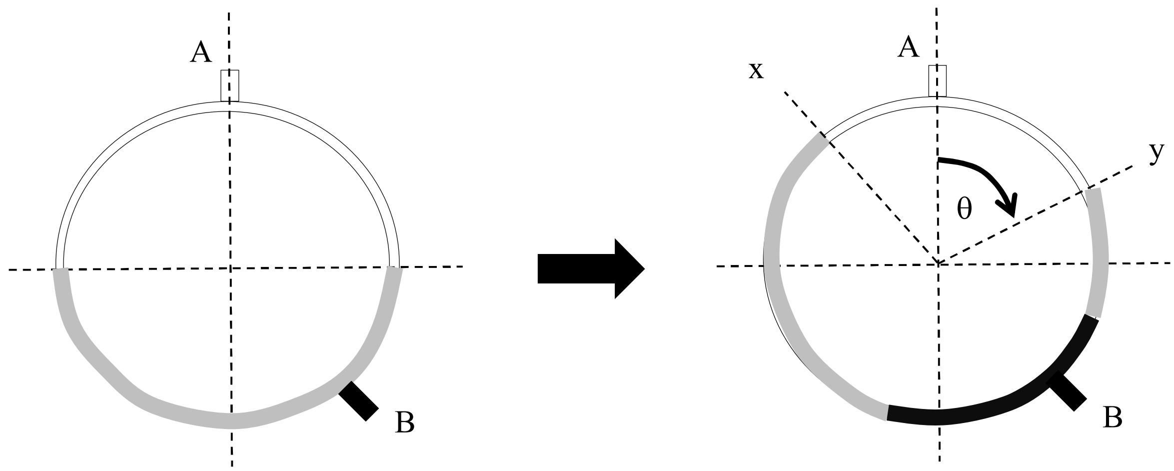
$$\frac{\Delta R_1}{R_1} = \frac{\sigma_1 - \nu\sigma_2}{E} S_g$$

$$\frac{\Delta R_2}{R_2} = \frac{\sigma_2 - \nu\sigma_1}{E} S_g$$

$$E_0 = \frac{r}{(1+r)^2}\left[\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2}\right] E_i$$

$$E_0 = \frac{1}{4}[81.23 + 0.25 \times 1.23 + 1.23 + 0.25 \times 81.23]\frac{3 \times 5}{150{,}000} = 2.6\ mV$$

Q.5 A circular tube with uniform small cross section area has two open gates; A at the top position (0°) and B at the lower right position (135° clockwise direction). If the tube is filled with water for the amount of a half of the total volume of the circular tube as shown in the left figure. When gate A is opened to the atmospheric pressure of 1 atm and at gate B is filled with mercury for the amount of a quarter of the total volume of the circular tube. Determine the value of θ in ° at the steady state as shown in the right figure.. (20)



Solution

The amount of water on the left occupies 135°. The amount of mercury occupies 90°. The amount of water on the right occupies 45°.

Equate pressure at x and y.

$$\rho_{water} g\big(r\cos(\theta) + r\sin(\theta - 45)\big) + \rho_{mercury} g\big(r\sin(135 - \theta) - r\sin(\theta - 45)\big) - \rho_{water} g\big(r\sin(135 - \theta) + r\sin(\theta)\big) = 0$$

$$\rho_{water}\big(\cos(\theta) + \sin(\theta - 45) - \sin(135 - \theta) - \sin(\theta)\big) + \rho_{mercury}\big(\sin(135 - \theta) - \sin(\theta - 45)\big) = 0$$

$$\rho_{water}\left(\cos(\theta) + \frac{1}{\sqrt{2}}\sin(\theta) - \frac{1}{\sqrt{2}}\cos(\theta) - \frac{1}{\sqrt{2}}\cos(\theta) - \frac{1}{\sqrt{2}}\sin(\theta) - \sin(\theta)\right) + \rho_{mercury}\left(\frac{1}{\sqrt{2}}\cos(\theta) + \frac{1}{\sqrt{2}}\sin(\theta) - \frac{1}{\sqrt{2}}\sin(\theta) + \frac{1}{\sqrt{2}}\cos(\theta)\right) = 0$$

$$\rho_{water}\left(\big(1 - \sqrt{2}\big)\cos(\theta) - \sin(\theta)\right) + \rho_{mercury}\left(\sqrt{2}\cos(\theta)\right) = 0$$

$$\tan(\theta) = \frac{\big(1 - \sqrt{2}\big)\rho_{water} + \sqrt{2}\rho_{mercury}}{\rho_{water}}$$

$$\tan(\theta) = \frac{\big(1 - \sqrt{2}\big) \times 1000 + \sqrt{2} \times 13550}{1000}$$

$$\theta = 86.95°$$